A NOTE ON THE REPRESENTATION OF POSITIVE POLYNOMIALS WITH STRUCTURED SPARSITY

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Abstract. We consider real polynomials in finitely many variables. Let the variables consist of finitely many blocks that are allowed to overlap in a certain way. Let the solution set of a finite system of polynomial inequalities be given where each inequality involves only variables of one block. We investigate polynomials that are positive on such a set and sparse in the sense that each monomial involves only variables of one block. In particular, we derive a short and direct proof for Lasserre's theorem on the existence of sums of squares certificates respecting the block structure. The motivation for the results can be found in the literature on numerical methods for global optimization of polynomials that exploit sparsity.

1. Introduction

Let $\mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \dots, X_n]$ be the ring of real polynomials in n variables $\bar{X} := (X_1, \dots, X_n)$. A subset $M \subseteq \mathbb{R}[\bar{X}]$ is called a *quadratic module* of $\mathbb{R}[\bar{X}]$ if it contains 1 and is closed under addition and multiplication with squares, i.e., $1 \in M$, $M + M \subseteq M$ and $\mathbb{R}[\bar{X}]^2 M \subseteq M$. A quadratic module $M \subseteq \mathbb{R}[\bar{X}]$ is called *archimedean* if $N - \sum_{i=1}^n X_i^2 \in M$ for some $N \in \mathbb{N}$.

Let $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]$ be given and set $g_0 := 1 \in \mathbb{R}[\bar{X}]$. These polynomials define a *basic closed semialgebraic set*

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\} \subseteq \mathbb{R}^n$$

and generate in $\mathbb{R}[\bar{X}]$ the quadratic module

$$M := \left\{ \sum_{i=0}^m \sigma_i g_i \mid \sigma_0, \dots, \sigma_m \in \sum \mathbb{R}[\bar{X}]^2 \right\} \subseteq \mathbb{R}[\bar{X}]$$

where $\sum \mathbb{R}[\bar{X}]^2$ denotes the set of sums of squares in the ring $\mathbb{R}[\bar{X}]$. Obviously, every polynomial from M is nonnegative on S. In particular, if M is archimedean, then S is compact. In 1991, Schmüdgen [S] proved a very surprising partial converse to these facts (for example, if $MM \subseteq M$, then M is archimedean if and only if S is compact). See [Sch, Section 1] and [PD] for this and a discussion when M is archimedean. Shortly after this groundbreaking work of Schmüdgen, Putinar [Put] proved the following theorem (note that $f > 0$ on S should mean that f is *strictly* positive on S):

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Theorem 1 (Putinar). *Suppose M is archimedean. Then for every $f \in \mathbb{R}[\bar{X}]$,*

$$f > 0 \text{ on } S \implies f \in M.$$

A very special case of this is the following theorem of Cassier [Cas, Théorème 4] (which can even be derived from earlier results like [Kri, Théorème 12]).

Corollary 2 (Cassier). *For all $R > 0$ and $f \in \mathbb{R}[\bar{X}]$, if $f > 0$ on the closed ball centered at the origin of radius R, then there are $\sigma, \tau \in \sum \mathbb{R}[\bar{X}]^2$ such that $f = \sigma + \tau(R^2 - \sum_{i=1}^n X_i^2)$.*

In 2001, Lasserre [L1] recognized that Putinar's result can be used for numerical computation of the minimum of a polynomial on a non-empty *compact* basic closed semialgebraic S by semidefinite programming (a well-known generalization of linear programming). The idea is to maximize λ such that $f - \lambda$ lies in a certain finite-dimensional subset of M that can be expressed in a semidefinite program (SDP for short) whose size depends on the size of the chosen subset of M. Since M can be exhausted by such subsets, the results of Schmüdgen and Putinar say that the accuracy of this method is arbitrarily good for large SDPs [L1, Sch].

In many problems, the polynomials f and g_i are sparse. Waki, Kim, Kojima and Muramatsu tried to take advantage of this sparsity and implemented corresponding SDP relaxations that turned out to be very efficient in practice [W]. They correspond to a certain subset of the quadratic module M reflecting the sparsity pattern of f and the g_i. But a theoretical result on the accuracy of such "sparse" SDP relaxations was only given recently by Lasserre [L2]. Lasserre's proof uses SDP duality, compactness arguments, theorems about the construction of measures with given marginals and Putinar's solution to the moment problem (a dual result to Theorem 1). Moreover, Lasserre assumes that S has non-empty interior. For general compact S, the result follows from an additional argument given by Kojima and Muramatsu [KM]. Recently, Kuhlmann and Putinar found a different proof for the result, see [KP].

In this note, we give a short direct proof of Lasserre's result taking only Corollary 2 for granted.

2. The Theorem

For $I \subseteq \{1, \dots, n\}$, we denote by X_I the set of variables $\{X_i \mid i \in I\}$ and by $\mathbb{R}[X_I]$ the polynomial ring in these variables. By $\mathbb{R}^I$ we denote the subspace of $\mathbb{R}^n$ which corresponds to the variables in X_I. In particular, an element of $\mathbb{R}[X_I]$ can be seen as a function on $\mathbb{R}^I$.

Following [L2], we suppose from now on that we have non-empty $I_1, \dots, I_r \subseteq \{1, \dots, n\}$ satisfying the following *running intersection property:*

(RIP) For all $i = 2, \dots, r$, there is $k < i$ such that $I_i \cap \bigcup_{j<i} I_j \subseteq I_k$.

Note that (RIP) is always satisfied if $r \in \{1, 2\}$. In case $r = 1$, our result will just be Putinar's Theorem 1 (without sparsity).

Lemma 3. *Let $K \subseteq \mathbb{R}$ be compact. Suppose $f = f_1 + \ldots + f_r$ with $f_j \in \mathbb{R}[X_{I_j}]$ and $f > 0$ on K^n. Then*

$$f = h_1 + \ldots + h_r$$

for some $h_j \in \mathbb{R}[X_{I_j}]$ with $h_j > 0$ on K^{I_j}.

Proof. The proof is by induction on r. The case $r = 1$ is trivial, now suppose $r = 2$. We may of course assume $K \neq \emptyset$. There is some $\varepsilon > 0$ such that $f = f_1 + f_2 \geq \varepsilon$ on K^n. Now consider the function $h : K^{I_1 \cap I_2} \to \mathbb{R}$ defined by

$$h(y) = \min\{f_1(x, y) \mid x \in K^{I_1 \setminus I_2}\} - \frac{\varepsilon}{2}.$$

Note that $K^{\emptyset}$ is a singleton and thus $h + \frac{\varepsilon}{2}$ is the minimum of f_1 on K^{I_1} in the case $I_1 \cap I_2 = \emptyset$. To show that h is continuous, consider $y, y' \in K^{I_1 \cap I_2}$. Choose $x, x' \in K^{I_1 \setminus I_2}$ minimizing $f(x, y)$ and $f(x', y')$, respectively. Then

$$\begin{aligned} |h(y) - h(y')| &= |f_1(x, y) - f_1(x', y')| \\ &\leq \max\{|f_1(x, y) - f_1(x, y')|, |f_1(x', y) - f_1(x', y')|\} \end{aligned}$$

shows that h is uniformly continuous because f_1 is uniformly continuous on the compact set K^{I_1}. We now claim that

$$f_1 - h \geq \frac{\varepsilon}{2} \text{ on } K^{I_1} \qquad \text{and} \qquad f_2 + h \geq \frac{\varepsilon}{2} \text{ on } K^{I_2}.$$

The first claim is clear by the definition of h. To show the second, suppose $(y, z) \in K^{I_2} = K^{I_1 \cap I_2} \times K^{I_2 \setminus I_1}$, choose $x \in K^{I_1 \setminus I_2}$ with $h(y) = f_1(x, y)$ and observe

$$f_2(y, z) + h(y) = f_2(y, z) + f_1(x, y) - \frac{\varepsilon}{2} = f(x, y, z) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}.$$

Now approximate h by a polynomial $p \in \mathbb{R}[X_{I_1 \cap I_2}]$ such that $|h - p| \leq \frac{\varepsilon}{4}$ on $K^{I_1 \cap I_2}$. Then

$$h_1 := f_1 - p > 0 \text{ on } K^{I_1} \qquad \text{and} \qquad h_2 := f_2 + p > 0 \text{ on } K^{I_2}$$

and $f = h_1 + h_2$. This completes the proof for $r = 2$.

For the induction step, suppose $r \geq 3$. Setting

$$\tilde{f} := f_1 + \ldots + f_{r-1} \in \mathbb{R}[X_{\bigcup_{j<r} I_j}],$$

we have $f = \tilde{f} + f_r$. Now the proof for the case $r = 2$ showed that there is some polynomial p, using only the variables indexed in $I_r \cap \bigcup_{j<r} I_j$, such that $\tilde{f} - p$ and $f_r + p$ are positive on appropriate cartesian powers of K. By (RIP), $p \in \mathbb{R}[X_{I_k}]$ for some $k < r$. Hence $\tilde{f} - p$ is a sum of polynomials from the rings $\mathbb{R}[X_{I_j}], j = 1, \ldots, r - 1$. The induction hypothesis therefore applies to $\tilde{f} - p$ and yields the result. $\square$

Again following [L2], we now suppose that for $j = 1, \ldots, r$, we have polynomials $g_1^{(j)}, \ldots, g_{l_j}^{(j)} \in \mathbb{R}[X_{I_j}]$ which define sets

$$S_j := \{x \in \mathbb{R}^{I_j} \mid g_1^{(j)}(x) \geq 0, \ldots, g_{l_j}^{(j)}(x) \geq 0\} \subseteq \mathbb{R}^{I_j}.$$

We do not require these sets to be compact at the moment. Let $S \subseteq \mathbb{R}^n$ be the basic closed semialgebraic set defined by all these polynomials $g_i^{(j)}$ in $\mathbb{R}^n$.

Lemma 4. *Suppose $f = f_1 + \ldots + f_r$ with $f_j \in \mathbb{R}[X_{I_j}]$ and $f > 0$ on S. Then for any bounded set $C \subseteq \mathbb{R}^n$, there are $0 < \lambda \leq 1$, $k \in \mathbb{N}$ and polynomials $h_j \in \mathbb{R}[X_{I_j}]$ with $h_j > 0$ on C such that*

$$f = \sum_{j=1}^{r} \sum_{i=1}^{l_j} (1 - \lambda g_i^{(j)})^{2k} g_i^{(j)} + \sum_{j=1}^{r} h_j.$$

Proof. Choose a compact set $K \subset \mathbb{R}$ such that $C \subseteq K^n$. Choose $0 < \lambda \leq 1$ such that $\lambda g_i^{(j)} \leq 1$ on K^n for all i, j. Set

$$f_k := f - \sum_{j=1}^{r} \sum_{i=1}^{l_j} \left(1 - \lambda g_i^{(j)}\right)^{2k} g_i^{(j)} \qquad (k \in \mathbb{N}).$$

We have $f_k \leq f_{k+1}$ on K^n for $k \in \mathbb{N}$ and one checks that for all $x \in K^n$ there exists $k \in \mathbb{N}$ such that $f_k(x) > 0$ (in fact $f_k \to f$ pointwise on $K^n \cap S$ and $f_k \to \infty$ pointwise on $K^n \setminus S$). By compactness of K^n, we find therefore $k \in \mathbb{N}$ with $f_k > 0$ on K^n. Now apply Lemma 3 to f_k. □

For S with non-empty interior, the following is Lasserre's result [L2, Corollary 3.9]. It has been extended to general compact S by Kojima and Muramatsu in [KM, Theorem 1].

Theorem 5 (Lasserre, Kojima, Muramatsu)**.** *Let the quadratic modules M_j generated by $g_1^{(j)}, \dots, g_{l_j}^{(j)}$ in $\mathbb{R}[X_{I_j}]$ be archimedean. If $f \in \mathbb{R}[X_{I_1}] + \dots + \mathbb{R}[X_{I_r}]$ and $f > 0$ on S, then $f \in M_1 + \dots + M_r$.*

Proof. Choose $R > 0$ big enough such that $R^2 - \sum_{i \in I_j} X_i^2 \in M_j$ for all j. Define $C \subseteq \mathbb{R}^n$ to be the closed ball with radius R around the origin. Now apply Lemma 4 and note that $h_j \in M_j$ by Corollary 2. □

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