CONNES' EMBEDDING CONJECTURE AND SUMS OF HERMITIAN SQUARES

IGOR KLEP AND MARKUS SCHWEIGHOFER

Abstract. We show that Connes' embedding conjecture on von Neumann algebras is equivalent to the existence of certain algebraic certificates for a polynomial in noncommuting variables to satisfy the following nonnegativity condition: The trace is nonnegative whenever self-adjoint contraction matrices of the same size are substituted for the variables. These algebraic certificates involve sums of hermitian squares and commutators. We prove that they always exist for a similar nonnegativity condition where elements of separable II_1-factors are considered instead of matrices. Under the presence of Connes' conjecture, we derive degree bounds for the certificates.

1. Introduction

The following has been conjectured in 1976 by Alain Connes [Con, Section V, pp. 105–107] in his paper on the classification of injective factors.

Conjecture 1.1 (Connes)**.** *If ω is a free ultrafilter on $\mathbb{N}$ and $\mathcal{F}$ is a separable II_1-factor, then $\mathcal{F}$ can be embedded into the ultrapower $\mathcal{R}^\omega$.*

We now explain the notation used in this conjecture. Set $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. If $(a_k)_{k\in\mathbb{N}}$ is a sequence in a Hausdorff space E and ω is an ultrafilter on $\mathbb{N}$, then $\lim_{k\to\omega} a_k = a$ means that $\{k \in \mathbb{N} \mid a_k \in U\} \in \omega$ for every neighborhood U of a. Such a limit is always unique and for compact E it always exists. Our reference for von Neumann algebras is [Tak]. When we speak of a trace τ of a finite factor $\mathcal{F}$, we always mean its canonical center valued trace $\tau : \mathcal{F} \to \mathbb{C}$ [Tak, Definition V.2.7]. Such a trace gives rise to the Hilbert-Schmidt norm on $\mathcal{F}$ given by $\|a\|_2^2 := \tau(a^*a)$ for $a \in \mathcal{F}$. This norm induces on $\mathcal{F}$ a topology which coincides on bounded sets with the strong operator topology. Let $\mathcal{R}$ denote the hyperfinite II_1-factor and τ_0 its trace. Consider the C^*-algebra $\ell^\infty(\mathcal{R}) := \{(a_k)_{k\in\mathbb{N}} \in \mathcal{R}^\mathbb{N} \mid \sup_{k\in\mathbb{N}} \|a_k\| < \infty\}$ (endowed with the supremum norm). Every ultrafilter ω on $\mathbb{N}$ defines a closed ideal $I_\omega := \{(a_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathcal{R}) \mid \lim_{k\to\omega} \|a_k\|_2 = 0\}$ in $\ell^\infty(\mathcal{R})$ and gives rise to the *ultrapower* $\mathcal{R}^\omega := \ell^\infty(\mathcal{R})/I_\omega$ (the quotient C^*-algebra) which is again a II_1-factor with trace $\tau_{0,\omega} : (a_k)_{k\in\mathbb{N}} + I_\omega \mapsto \lim_{k\to\omega} \tau_0(a_k)$. By an *embedding* of $\mathcal{F}$ into $\mathcal{R}^\omega$, we always mean a trace preserving $*$-homomorphism.

Recent work of Kirchberg [Kir] shows that Connes' conjecture has several equivalent reformulations in operator algebras and Banach space theory, among which

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is the statement that there exists a unique C^*-norm on the tensor product of the universal C^*-algebra of a free group with itself. Voiculescu [Voi] defines a notion of entropy in free probability theory whose behavior is intimately connected with Connes' conjecture. In this article, we show that Conjecture 1.1 is equivalent to a purely algebraic statement which resembles recently proved theorems on sums of squares representations of polynomials. Before presenting the algebraic reformulation, we need to introduce some notions.

Let always $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. As we will rarely need it, we denote the complex imaginary unit by $\mathbb{i}$ so that the letter i can be used as an index. We denote the complex conjugate of a complex number $c = a + \mathbb{i}b$ $(a, b \in \mathbb{R})$ by $c^* := a - \mathbb{i}b$.

We assume that all rings are associative, have a unit element and that ring homomorphisms preserve the unit element. Throughout the article, we assume that $n \in \mathbb{N}$ and $\bar{X} := (X_1, \dots, X_n)$ are variables (or symbols). We write $\langle \bar{X} \rangle$ for the monoid freely generated by $\bar{X}$, i.e., $\langle \bar{X} \rangle$ consists of *words* in the n letters $X_1, \dots, X_n$ (including the empty word denoted by 1). For any commutative ring R, let $R\langle \bar{X} \rangle$ denote the associative R-algebra freely generated by $\bar{X}$, i.e., the elements of $R\langle \bar{X} \rangle$ are polynomials in the noncommuting variables $\bar{X}$ with coefficients in R. An element of the form aw where $0 \neq a \in R$ and $w \in \langle \bar{X} \rangle$ is called a *monomial* and a its *coefficient*. Hence words are monomials whose coefficient is 1. Write $R\langle \bar{X} \rangle_k$ for the R-submodule consisting of the polynomials of degree at most k and $\langle \bar{X} \rangle_k$ for the set of words $w \in \langle \bar{X} \rangle$ of length at most k.

Definition 1.2. Let R be a commutative ring. Two polynomials $f, g \in R\langle \bar{X} \rangle$ are called *cyclically equivalent* ($f \overset{\text{cyc}}{\sim} g$) if $f - g$ is a sum of commutators in $R\langle \bar{X} \rangle$.

The following remark shows that cyclic equivalence can easily be checked and that it is "stable" under ring extensions in the following sense: Given an extension of commutative rings $R \subseteq R'$ and $f, g \in R\langle \bar{X} \rangle$, then $f \overset{\text{cyc}}{\sim} g$ in $R\langle \bar{X} \rangle$ if and only if $f \overset{\text{cyc}}{\sim} g$ in $R'\langle \bar{X} \rangle$.

Remark 1.3. Let R be a commutative ring.

(a) For $v, w \in \langle \bar{X} \rangle$, we have $v \overset{\text{cyc}}{\sim} w$ if and only if there are $v_1, v_2 \in \langle \bar{X} \rangle$ such that $v = v_1 v_2$ and $w = v_2 v_1$.

(b) Two polynomials $f = \sum_{w \in \langle \bar{X} \rangle} a_w w$ and $g = \sum_{w \in \langle \bar{X} \rangle} b_w w$ $(a_w, b_w \in R)$ are cyclically equivalent if and only if for each $v \in \langle \bar{X} \rangle$,

$$\sum_{\substack{w \in \langle \bar{X} \rangle \\ w \overset{\text{cyc}}{\sim} v}} a_w = \sum_{\substack{w \in \langle \bar{X} \rangle \\ w \overset{\text{cyc}}{\sim} v}} b_w.$$

We call a map $a \mapsto a^*$ on a ring R an *involution* if $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$ and $a^{**} = a$ for all $a, b \in R$. If $*$ is an involution on R (e.g. complex conjugation on $\mathbb{C}$ or the identity on $\mathbb{R}$), then we extend $*$ to *the* involution on $R\langle \bar{X} \rangle$ such that $X_i^* = X_i$. For each word $w \in \langle \bar{X} \rangle$, w^* is its reverse.

Definition 1.4. Let R be a ring with involution $*$. For each subset $S \subseteq R$, we introduce the set

$$\operatorname{Sym} S := \{g \in S \mid g^* = g\}$$

of its *symmetric elements*. Elements of the form $g^* g$ $(g \in R)$ are called *hermitian squares*. A subset $M \subseteq \operatorname{Sym} R$ is called a *quadratic module* if $1 \in M$, $M + M \subseteq M$ and $g^* M g \subseteq M$ for all $g \in R$.

We can now state the algebraic reformulation of the conjecture.

Conjecture 1.5 (Algebraic version of Connes' conjecture)**.** *Suppose $f \in \Bbbk\langle \bar{X}\rangle$. If $\Bbbk = \mathbb{R}$, assume moreover that $f = f^*$. Then the following are equivalent:*

(i) $\operatorname{tr}(f(A_1, \dots, A_n)) \geq 0$ *for all $s \in \mathbb{N}$ and self-adjoint contractions $A_i \in \Bbbk^{s\times s}$;*

(ii) *For every $\varepsilon \in \mathbb{R}_{>0}$, $f+\varepsilon$ is cyclically equivalent to an element in the quadratic module generated by $1 - X_i^2$ $(1 \leq i \leq n)$ in $\Bbbk\langle \bar{X}\rangle$.*

Theorem 1.6. *The following are equivalent:*

(i) *Connes' embedding conjecture* 1.1 *holds;*

(ii) *The algebraic version* 1.5 *of Connes' embedding conjecture holds;*

(iii) *The implication* (i)⇒(ii) *from Conjecture* 1.5 (for $\Bbbk = \mathbb{R}$) *holds for all $n \in \mathbb{N}$ and $f \in \operatorname{Sym}\mathbb{R}\langle \bar{X}\rangle$.*

This theorem will be proved in Section 3. Reformulations of Connes' conjecture that involve sums of squares have already been given by Hadwin [Had] and Rădulescu [R2]. However, Hadwin works with elements of a certain C^*-algebra and Rădulescu with certain power series instead of polynomials. In addition, both work with *limits* of sums of squares. The advantage of our Conjecture 1.5 is that it is purely algebraic and therefore reveals the analogy to previously proved theorems on sums of squares representations of polynomials.

Looking for a counterpart of Conjecture 1.5 for the ring $\mathbb{R}[\bar{X}]$ of polynomials in *pairwise commuting* variables, we replace cyclic equivalence by equality and take the identity involution. Furthermore, in condition (i), the matrices A_i should now be assumed to commute pairwise. But then they can be simultaneously diagonalized. One therefore arrives naturally at the following statement which is a particular case of Putinar's theorem [Put] (we work here over $\Bbbk = \mathbb{R}$ since a complex polynomial which is real on $[-1, 1]^n$ has automatically real coefficients).

Theorem 1.7 (Putinar)**.** *For every $f \in \mathbb{R}[\bar{X}]$, the following are equivalent:*

(i) *$f \geq 0$ on $[-1, 1]^n$;*

(ii) *For all $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ lies in the quadratic module generated by $1 - X_i^2$ in $\mathbb{R}[\bar{X}]$ endowed with the trivial involution.*

For noncommuting variables, one can also consider equality instead of cyclic equivalence. The natural counterpart to Conjecture 1.5 is then the following particular case of [HM, Theorem 1.2] (we have omitted the hypothesis $f = f^*$ which is redundant by [KS, Proposition 2.3]). For some related results see also [Cim, KS].

Theorem 1.8 (Helton, McCullough)**.** *The following are equivalent for $f \in \Bbbk\langle \bar{X}\rangle$:*

(i) *$f(A_1, \dots, A_n)$ is positive semidefinite for all $s \in \mathbb{N}$ and self-adjoint contractions $A_i \in \Bbbk^{s\times s}$;*

(ii) *For all $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ lies in the quadratic module generated by $1 - X_i^2$ in $\Bbbk\langle \bar{X}\rangle$.*

The paper is organized as follows. Section 2 deals with polynomials whose trace is not only nonnegative but vanishes. We prove that these polynomials are sums of commutators. This result is needed subsequently as a tool. The objective of Section 3 is to prove Theorem 1.6. Along the way, we obtain for example that Conjecture 1.5 holds when matrices are replaced by elements of II_1-factors (see Theorem 3.12). In Section 4, we show that Putinar's Theorem 1.7 implies Conjecture 1.5 for certain polynomials in two variables. Finally, in Section 5 we establish the existence of certain degree bounds for Conjecture 1.5.

2. Polynomials with vanishing trace

Theorem 2.1. *Let $d \in \mathbb{N}$ and $f \in \Bbbk\langle \bar{X} \rangle_d$ satisfy*

$$\operatorname{tr}(f(A_1, \dots, A_n)) = 0 \tag{1}$$

for all self-adjoint contractions $A_1, \dots, A_n \in \Bbbk^{d \times d}$. In the case $\Bbbk = \mathbb{R}$, assume moreover that $f = f^*$. Then $f \overset{\text{cyc}}{\sim} 0$.

Proof. We call a polynomial $(k_1, \dots, k_n)$-multihomogeneous ($k_i \in \mathbb{N}_0$) if each of its monomials has for all i degree k_i with respect to the variable X_i. The $(k_1, \dots, k_n)$-multihomogeneous part of a polynomial is the sum of all its $(k_1, \dots, k_n)$-multihomogeneous monomials. Every polynomial is the sum of its multihomogeneous parts. The multihomogeneous parts of a symmetric polynomial are symmetric. We start by proving the following reduction step which will be used several times during the proof.

Reduction step. If $f \in \Bbbk\langle \bar{X} \rangle$ satisfies (1) for all self-adjoint contractions $A_1, \dots, A_n \in \Bbbk^{d \times d}$, then all its multihomogeneous parts g satisfy

$$\operatorname{tr}(g(A_1, \dots, A_n)) = 0 \tag{2}$$

for all self-adjoint (not necessarily contraction) matrices $A_1, \dots, A_n \in \Bbbk^{d \times d}$.

Proof of the reduction step. Fix self-adjoint contractions $A_1, \dots, A_n \in \Bbbk^{d \times d}$. Then for every $\lambda \in \mathbb{R}$ with $|\lambda| \le 1$, the matrix λA_1 is again a self-adjoint contraction and (1) implies $\operatorname{tr}(f(\lambda A_1, A_2, \dots, A_n)) = 0$. But the latter expression defines a complex polynomial in λ where the coefficient belonging to λ^k is $\operatorname{tr}(g_k(A_1, \dots, A_n))$ where $g_k \in \Bbbk\langle \bar{X} \rangle$ is the sum of all monomials of f having degree k with respect to X_1. Since this polynomial vanishes at infinitely many points λ, all its coefficients must be zero. This shows that $\operatorname{tr}(g_k(A_1, \dots, A_n)) = 0$ for all self-adjoint contractions $A_1, \dots, A_n \in \Bbbk^{d \times d}$. We are therefore reduced to the case where each f is homogeneous in X_1. Now repeat exactly the same arguments for the other variables. In this way, we see that (2) holds for all multihomogeneous parts g of f and all self-adjoint contraction matrices $A_i \in \Bbbk^{d \times d}$.

As a first application of the now justified reduction step, we see that our hypothesis implies that (1) holds for all self-adjoint (not necessarily contraction) matrices. Hence it suffices to show the following claim for all $k \in \mathbb{N}$ by induction on k.

Claim. For all $n, d \in \mathbb{N}$ and $f \in \Bbbk\langle X_1, \dots, X_n \rangle_d$ (with $f = f^*$ if $\Bbbk = \mathbb{R}$) having degree at most k in each individual variable X_i and satisfying (1) for all self-adjoint $A_1, \dots, A_n \in \Bbbk^{d \times d}$, we have $f \overset{\text{cyc}}{\sim} 0$.

Induction basis $k = 1$. By the above reduction step and by forgetting the variables not appearing in f, we may assume that f is $(1, \dots, 1)$-homogeneous (also called multilinear), i.e., each variable appears in each monomial of f exactly once. This means that f can be written as $f = \sum_{\sigma \in S_n} a_\sigma X_{\sigma(1)} \cdots X_{\sigma(n)}$ where S_n is the symmetric group on $\{1, \dots, n\}$ and $a_\sigma \in \Bbbk$ for all $\sigma \in S_n$. By the definition of cyclic equivalence, we have to show that for each $\tau \in S_n$, the sum over all a_σ such that $X_{\sigma(1)} \cdots X_{\sigma(n)}$ equals one of the n monomials

$$X_{\tau(1)} \cdots X_{\tau(n)}, \quad X_{\tau(2)} \cdots X_{\tau(n)} X_{\tau(1)}, \quad \dots, \quad X_{\tau(n)} X_{\tau(1)} \cdots X_{\tau(n-1)}$$

is zero. By renumbering the variables $\bar{X}$, we may without loss of generality assume that τ is the identity permutation. Let $E_{i,j} \in \Bbbk^{d \times d}$ be the matrix with all entries zero except for a one in the i-th row and j-th column. Note that $E_{i,j} E_{k,\ell} = \delta_{j,k} E_{i,\ell}$

and $E_{i,j} + E_{j,i}$ is self-adjoint. Then it follows from the multilinearity of f that

$$\begin{aligned}0 &= \operatorname{tr}(f(E_{1,2} + E_{2,1}, E_{2,3} + E_{3,2}, \dots, E_{n-1,n} + E_{n,n-1}, E_{n,1} + E_{1,n}))\\ &= \operatorname{tr}(f(E_{1,2}, E_{2,3}, \dots, E_{n-1,n}, E_{n,1})) + \dots + \operatorname{tr}(f(E_{2,1}, E_{3,2}, \dots, E_{n,n-1}, E_{1,n}))\end{aligned}$$

where the sum in the last line has 2^n terms. Each of the $2^n - 2$ terms represented by the dots must vanish. This corresponds to the fact that the only paths on the cyclic graph with n nodes passing through each of the n edges exactly once are those paths that go through each edge with the same orientation (either "clockwise" $i \mapsto i+1$ or "counterclockwise" $i \mapsto i-1$ modulo n). There are only $2n$ such paths which are determined by their starting point and their orientation. The n clockwise paths show that the first of the 2^n terms is the sum of those a_σ such that $X_{\sigma(1)} \cdots X_{\sigma(n)}$ equals one of the monomials

$$X_1 \cdots X_n, \quad X_2 \cdots X_n X_1, \quad \dots, \quad X_n X_1 \cdots X_{n-1}. \tag{3}$$

Calling this sum a, we see that $a = 0$ is exactly what we have to show. The n counterclockwise paths show that the last of the 2^n terms is the sum b of those a_σ such that $X_{\sigma(1)} \cdots X_{\sigma(n)}$ equals one of the monomials

$$X_n \cdots X_1, \quad X_{(n-1)} \cdots X_1 X_n, \quad \dots, \quad X_1 X_n \cdots X_2$$

which are just the monomials arising from (3) by applying the involution $*$. Hence $0 = a + b$. In the case $\Bbbk = \mathbb{R}$, we use the hypothesis $f = f^*$, to see that $a = b$ and therefore $a = 0$ as desired. In the case $\Bbbk = \mathbb{C}$, additional work is needed. Choose $\zeta \in \mathbb{C}$ such that $\zeta^n = \mathbb{i}$. Using similar arguments as above, we get

$$\begin{aligned}0 &= \operatorname{tr}(f(\zeta E_{1,2} + \zeta^* E_{2,1}, \zeta E_{2,3} + \zeta^* E_{3,2}, \dots, \zeta E_{n-1,n} + \zeta^* E_{n,n-1}, \zeta E_{n,1} + \zeta^* E_{1,n}))\\ &= \zeta^n \operatorname{tr}(f(E_{1,2}, E_{2,3}, \dots, E_{n,1})) + \dots + (\zeta^*)^n \operatorname{tr}(f(E_{2,1}, E_{3,2}, \dots, E_{1,n}))\\ &= \mathbb{i} \operatorname{tr}(f(E_{1,2}, E_{2,3}, \dots, E_{n,1})) - \mathbb{i} \operatorname{tr}(f(E_{2,1}, E_{3,2}, \dots, E_{1,n}))\\ &= \mathbb{i}a - \mathbb{i}b = \mathbb{i}(a-b)\end{aligned}$$

which together with $a + b = 0$ yields $a = 0$.

Induction step from $k-1$ to k ($k \geq 2$). By the above reduction step, we can assume that f is $(k_1, \dots, k_n)$-multihomogeneous where $k_1 = \dots = k_m = k$ and $k_i < k$ for all $i \in \{m+1, \dots, n\}$. We assume $m \geq 1$ since otherwise the induction hypothesis applies immediately. Now we define recursively a finite sequence $f_0, f_1, \dots, f_m$ of polynomials

$$f_i \in \Bbbk\langle X_1, X_1', \dots, X_i, X_i', X_{i+1}, X_{i+2} \dots, X_n\rangle$$

by $f_0 := f$ and

$$\begin{aligned}f_i :=& f_{i-1}(X_1, X_1', X_2, X_2', \dots, X_{i-1}, X_{i-1}', X_i + X_i', X_{i+1}, \dots, X_n)\\ &- f_{i-1}(X_1, X_1', X_2, X_2', \dots, X_{i-1}, X_{i-1}', X_i, X_{i+1}, \dots, X_n)\\ &- f_{i-1}(X_1, X_1', X_2, X_2', \dots, X_{i-1}, X_{i-1}', X_i', X_{i+1}, \dots, X_n).\end{aligned}$$

In other words, each monomial of f_{i-1} gives rise to the $2^k - 2$ monomials of f_i which are obtained by replacing at least one but not all of the occurrences of X_i by X_i'. It is important to note that f_{i-1} can be retrieved from f_i by resubstituting $X_i' \mapsto X_i$, more exactly

$$f_{i-1} = \frac{1}{2^k - 2} f_i(X_1, X_1', X_2, X_2', \dots, X_{i-1}, X_{i-1}', X_i, X_i, X_{i+1}, X_{i+2}, \dots, X_n) \tag{4}$$

(we use here that $k \geq 2$). The polynomial f_m has degree at most $k-1$ with respect to each of its variables and we have $\operatorname{tr}(f_m(A_1, A_1', \ldots, A_m, A_m', A_{m+1}, \ldots, A_n)) = 0$ for all self-adjoint $A_i, A_i' \in \mathbb{k}^{d \times d}$. We now apply the induction hypothesis (for polynomials in $2m + (n-m)$ variables) to conclude that $f_m \overset{\text{cyc}}{\sim} 0$, i.e., f_m is a sum of commutators. Using (4), we get successively that f_{m-1}, f_{m-2}, $\ldots$, $f_0 = f$ are also sums of commutators and so $f \overset{\text{cyc}}{\sim} 0$. $\square$

Remark 2.2. For $\mathbb{k} = \mathbb{R}$, the assumption $f = f^*$ in Theorem 2.1 is indispensable as shown by $f := XYZ - ZYX \in \mathbb{R}\langle X, Y, Z\rangle$. For all $d \in \mathbb{N}$ and all self-adjoint $A, B, C \in \mathbb{R}^{d \times d}$, we have $\operatorname{tr}(f(A, B, C)) = 0$ but f is not cyclically equivalent to 0.

Proposition 2.3. *Let $d \in \mathbb{N}$ and $f \in \mathbb{C}\langle \bar{X}\rangle_d$ satisfy $\operatorname{tr}(f(A_1, \ldots, A_n)) \in \mathbb{R}$ for all self-adjoint contractions $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$. Then there is some g such that*

$$f \overset{\text{cyc}}{\sim} g \in \operatorname{Sym} \mathbb{R}\langle \bar{X}\rangle_d.$$

Proof. If f were not cyclically equivalent to $p := \frac{f+f^*}{2}$, then f would not be cyclically equivalent to f^*. But then Theorem 2.1 would yield complex self-adjoint contraction matrices $A_i \in \mathbb{C}^{d \times d}$ such that

$$\operatorname{tr}(f(A_1, \ldots, A_n)) \neq \operatorname{tr}(f^*(A_1, \ldots, A_n)) = \operatorname{tr}(f(A_1, \ldots, A_n))^*,$$

contradicting the hypothesis. Hence $f \overset{\text{cyc}}{\sim} p$. Write $p = g + \mathbb{i}h$ with $g, h \in \mathbb{R}\langle \bar{X}\rangle$. We have $g + \mathbb{i}h = p = p^* = (g + \mathbb{i}h)^* = g^* - \mathbb{i}h^*$ and hence $g = g^*$ (and $h = -h^*$). The "real trace condition" which is fulfilled for f by hypothesis, is also satisfied by p (since $p \overset{\text{cyc}}{\sim} f$) and g (because $g = g^*$) and therefore by $\mathbb{i}h$. But this is only possible if $\operatorname{tr}(h(A_1, \ldots, A_n)) = 0$ for all self-adjoint $A_i \in \mathbb{R}^{d \times d}$. Applying Theorem 2.1 again, we obtain $h \overset{\text{cyc}}{\sim} 0$. Thus $f \overset{\text{cyc}}{\sim} g \in \operatorname{Sym} \mathbb{R}\langle \bar{X}\rangle_d$. $\square$

3. Algebraic formulation of Connes' conjecture

Definition 3.1. We call a linear map $\varphi : \mathbb{k}\langle \bar{X}\rangle \to \mathbb{k}$ a *tracial contraction state* if
(a) $\varphi(fg) = \varphi(gf)$ for all $f, g \in \mathbb{k}\langle \bar{X}\rangle$;
(b) $|\varphi(w)| \leq 1$ for all $w \in \langle \bar{X}\rangle$;
(c) $\varphi(f^* f) \geq 0$ for all $f \in \mathbb{k}\langle \bar{X}\rangle$;
(d) $\varphi(1) = 1$;
(e) (redundant if $\mathbb{k} = \mathbb{C}$, see Remark 3.3 below) $\varphi(f^*) = \varphi(f)^*$ for all $f \in \mathbb{k}\langle \bar{X}\rangle$.

Example 3.2. If $A_1, \ldots, A_n \in \mathbb{k}^{s \times s}$ are self-adjoint contraction matrices, then

$$\varphi : \mathbb{k}\langle \bar{X}\rangle \to \mathbb{k}, \qquad f \mapsto \frac{1}{s} \operatorname{tr}(f(A_1, \ldots, A_n))$$

is a tracial contraction state.

Remark 3.3. If $\mathbb{k} = \mathbb{C}$, then (e) follows automatically from (a)–(d) in Definition 3.1. Indeed, it follows from (c) and the identity

$$f = \left(\frac{f+1}{2}\right)^2 - \left(\frac{f-1}{2}\right)^2 \tag{5}$$

that $\varphi(f) \in \mathbb{R}$ for $f \in \operatorname{Sym} \mathbb{C}\langle \bar{X}\rangle$. Now use that $\mathbb{C}\langle \bar{X}\rangle = \operatorname{Sym} \mathbb{C}\langle \bar{X}\rangle \oplus \mathbb{i} \operatorname{Sym} \mathbb{C}\langle \bar{X}\rangle$ as a real vector space which follows from the identity

$$f = \frac{f + f^*}{2} + \mathbb{i}\frac{f - f^*}{2\mathbb{i}}. \tag{6}$$

Remark 3.4. In Definition 3.1, condition (b) can equivalently be replaced by each of the following conditions:

(b') φ is a contraction with respect to the 1-norm on $\Bbbk\langle\bar{X}\rangle$ defined by

$$\Big\|\sum_{w\in\langle\bar{X}\rangle} a_w w\Big\|_1 := \sum_{w\in\langle\bar{X}\rangle} |a_w| \qquad (a_w \in \Bbbk,\ \text{only finitely many} \neq 0);$$

(b") The set $\{\varphi(X_i^{2k}) \mid k \in \mathbb{N}, 1 \leq i \leq n\}$ is bounded;

(b"') $\liminf_{k\to\infty} |\varphi(X_i^{2k})| < \infty$ for $i \in \{1,\dots,n\}$.

For details, consult [Had, Theorem 1.3].

Definition 3.5. For any commutative ring R with involution, we denote by $M_R^{(n)} \subseteq \operatorname{Sym} R\langle\bar{X}\rangle$ the quadratic module generated by $1-X_1^2,\dots,1-X_n^2$ in $R\langle\bar{X}\rangle$. Most of the time, there will be no doubt about the number n of variables and we will simply write M_R instead of $M_R^{(n)}$.

Remark 3.6. In any $\mathbb{Q}$-algebra R, the identity

$$1-a+\frac{1}{m}a^m = \frac{1}{m}+\frac{1}{m}(1-a)^2\sum_{k=0}^{m-2}(m-1-k)a^k$$

holds for all $m \in \mathbb{N}$ and $a \in R$.

Lemma 3.7. *In Definition* 3.1*, conditions* (b) *and* (c) *can be replaced by the condition* $\varphi(M_\Bbbk) \subseteq \mathbb{R}_{\geq 0}$.

Proof. Assume that $\varphi(M_\Bbbk) \subseteq \mathbb{R}_{\geq 0}$. Condition (c) follows immediately since the set of all hermitian squares is contained in $M_\Bbbk$. For $w \in \langle\bar{X}\rangle$, $\mu \in \Bbbk$ with $|\mu| = 1$, $s \in \mathbb{N}$ and self-adjoint contraction matrices $A_1,\dots,A_n \in \Bbbk^{s\times s}$,

$$\Big(1-\frac{\mu w+(\mu w)^*}{2}\Big)(A_1,\dots,A_n)$$

is positive semidefinite. Hence by Theorem 1.8, $1-\frac{\mu w+(\mu w)^*}{2}+\varepsilon \in M_\Bbbk$ for every $\varepsilon \in \mathbb{R}_{>0}$. This implies $\varphi(1-\frac{\mu w+(\mu w)^*}{2}) \geq 0$ and so $\operatorname{Re}(\mu\varphi(w)) = \operatorname{Re}\varphi(\mu w) \leq 1$. Since $\mu \in \Bbbk$ with $|\mu| = 1$ was arbitrary, this implies $|\varphi(w)| \leq 1$.

For the converse, let $g \in \Bbbk\langle\bar{X}\rangle$ be arbitrary. Then for every $m \in \mathbb{N}$,

$$\begin{aligned} g^*(1-X_i^2)g &= g^*(1-X_i^2+\frac{1}{m}X_i^{2m})g-\frac{1}{m}g^*X_i^{2m}g \\ &= g^*\Big(\frac{1}{m}+\frac{1}{m}(1-X_i^2)^2\sum_{k=0}^{m-2}(m-1-k)X_i^{2k}\Big)g-\frac{1}{m}g^*X_i^{2m}g \end{aligned}$$

by Remark 3.6. By applying φ to the last expression, the first summand becomes nonnegative by (c), while $\frac{1}{m}\varphi(g^*X_i^{2m}g)$ goes to zero when $m \to \infty$ since φ is continuous with respect to the 1-norm by (b). This proves that $\varphi(g^*(1-X_i^2)g) \geq 0$. Hence $\varphi(M_\Bbbk) \subseteq \mathbb{R}_{\geq 0}$. □

Definition 3.8. If R is a ring with involution $*$ and $M \subseteq \operatorname{Sym} R$ is a quadratic module, then we define its *ring of bounded elements*

$$H(M) := \{g \in R \mid N-g^*g \in M \text{ for some } N \in \mathbb{N}\}.$$

This is indeed a $*$-subring of R as proved in [Vid, Lemma 4].

In algebra, one says that a quadratic module $M \subseteq \operatorname{Sym} R$ is archimedean if $H(M) = R$. Unfortunately, this has a completely different meaning in the context of ordered vector spaces [Hol, p. 202, §22A]. We avoid this terminology and instead use the concept of algebraic interior (or core) points [Hol, p. 7, §2C].

Definition 3.9. Let V be a $\Bbbk$-vector space and $C \subseteq V$. A vector $v \in V$ is called an *algebraic interior point* of C if for each $u \in V$ there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $v + \lambda u \in C$ for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq \varepsilon$.

The following is well-known but so important for us that we give a proof of it.

Proposition 3.10. *If R is an $\mathbb{R}$-algebra and $M \subseteq \operatorname{Sym} R$ a quadratic module, then $H(M) = R$ if and only if 1 is an algebraic interior point of M in $\operatorname{Sym} R$.*

Proof. If 1 is an algebraic interior point of M in $\operatorname{Sym} R$ and $g \in R$, we find some $N \in \mathbb{N}$ such that $1 - \frac{1}{N} g^* g \in M$, i.e., $N - g^* g \in M$.

Conversely, suppose that $H(M) = R$ and let $u \in \operatorname{Sym} R$ be given. Then $u = \left(\frac{u+1}{2}\right)^2 - \left(\frac{u-1}{2}\right)^2$. Choose $N \in \mathbb{N}$ such that $N - \left(\frac{u-1}{2}\right)^2 \in M$ and set $\varepsilon := \frac{1}{N}$. Then $1 + \lambda u \in M$ for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq \varepsilon$. □

Lemma 3.11. *If R is a $*$-subfield of $\mathbb{C}$, then $H(M_R) = R\langle \bar{X} \rangle$.*

Proof. We have $R \subseteq H(M_R)$ and $1 - X_i^2 \in M_R$, hence $X_i \in H(M_R)$. Since $H(M_R)$ is a subring of $R\langle \bar{X} \rangle$, this implies $H(M_R) = R\langle \bar{X} \rangle$. □

Theorem 3.12. *For $f \in \mathbb{C}\langle \bar{X} \rangle$, the following are equivalent:*

(i) *$\tau(f(A_1, \ldots, A_n)) \geq 0$ for every separable II_1-factor $\mathcal{F}$ with trace τ and all self-adjoint contractions $A_1, \ldots, A_n \in \mathcal{F}$;*

(ii) *$\varphi(f) \geq 0$ for all tracial contraction states φ on $\mathbb{C}\langle \bar{X} \rangle$;*

(iii) *For every $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ is cyclically equivalent to an element of $M_{\mathbb{C}}$.*

Proof. It is immediate from Lemma 3.7 that (iii) implies (ii). It is trivial that (ii) implies (i). To see that (i) implies (iii), we proceed as follows. Suppose that there is $\varepsilon > 0$ such that $f + \varepsilon$ is not cyclically equivalent to an element of $M_{\mathbb{C}}$. We start by constructing a tracial contraction state L on $\mathbb{C}\langle \bar{X} \rangle$ such that $L(f) \notin \mathbb{R}$ or $L(f) < 0$.

If f is not cyclically equivalent to *any* symmetric element, then Proposition 2.3 yields a tracial contraction state $L : \mathbb{C}\langle \bar{X} \rangle \to \mathbb{C}$ coming from matrices (cf. Example 3.2) such that $L(f) \notin \mathbb{R}$.

If f is cyclically equivalent to a symmetric element of $\mathbb{C}\langle \bar{X} \rangle$, then we may assume without loss of generality that f *is* symmetric. Define $U := \{g \in \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle \mid g \overset{\text{cyc}}{\sim} 0\}$. Then $M_{\mathbb{C}} + U$ is a convex cone in the real vector space $\operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle$. By Lemma 3.11, 1 is an algebraic interior point of $M_{\mathbb{C}}$ and therefore of $M_{\mathbb{C}} + U$. Since $f + \varepsilon \notin M_{\mathbb{C}} + U$ and $M_{\mathbb{C}} + U$ possesses an algebraic interior point, we can apply the Eidelheit-Kakutani separation theorem [Hol, p. 15, §4B Corollary] to obtain an $\mathbb{R}$-linear functional $L_0 : \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle \to \mathbb{R}$ such that $L_0(M_{\mathbb{C}} + U) \subseteq \mathbb{R}_{\geq 0}$ and $L_0(f + \varepsilon) \in \mathbb{R}_{\leq 0}$. In particular, $L_0(U) = \{0\}$. Using (6), L_0 can be extended uniquely to a $\mathbb{C}$-linear functional L on $\mathbb{C}\langle \bar{X} \rangle$. Obviously, L is a state. To prove that L is tracial, let $g, h \in \mathbb{C}\langle \bar{X} \rangle$ be arbitrary and write $g = g_1 + \mathbb{i} g_2$ and $h = h_1 + \mathbb{i} h_2$ for $g_1, g_2, h_1, h_2 \in \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle$. Then $[g, h] = [g_1, h_1] + \mathbb{i}[g_2, h_1] + \mathbb{i}[g_1, h_2] - [g_2, h_2]$. The second and the third summand are symmetric commutators and are thus mapped to 0 by L. Similarly, $L([g_j, h_j]) = -\mathbb{i} L([\mathbb{i} g_j, h_j])) = 0$ for $j = 1, 2$. Thus $L([g, h]) = 0$, as desired.

In both cases we obtain a tracial contraction state L with $L(f) \notin \mathbb{R}_{\geq 0}$. (Note that this already proves (ii) $\Rightarrow$ (iii).)

Endow $\mathbb{C}\langle \bar{X}\rangle$ with the 1-norm defined in Remark 3.4. By the Banach-Alaoglu theorem [Hol, p. 70, §12D Corollary 1], the convex set of all tracial contraction states is weak $*$-compact. Thus by the Krein-Milman theorem [Hol, p. 74, §13B Theorem] we may assume that L is an extreme tracial contraction state.

We now apply the Gelfand-Naimark-Segal construction with L. By the Cauchy-Schwarz inequality for semi-scalar products, $N := \{p \in \mathbb{C}\langle \bar{X}\rangle \mid L(p^*p) = 0\}$ is a subspace of $\mathbb{C}\langle \bar{X}\rangle$. Similarly, we see that

$$\langle \overline{p}, \overline{q}\rangle := L(q^*p) \tag{7}$$

defines a scalar product on $\mathbb{C}\langle \bar{X}\rangle/N$, where $\overline{p} := p + N$ denotes the residue class of $p \in \mathbb{C}\langle \bar{X}\rangle$ modulo N. Let E denote the completion of $\mathbb{C}\langle \bar{X}\rangle/N$ with respect to this scalar product. Since $1 \notin N$, E is nontrivial. Observe that E is separable.

To prove that N is a left ideal of $\mathbb{C}\langle \bar{X}\rangle$, we fix $i \in \{1, \ldots, n\}$ and show that $X_iN \subseteq N$. Since $1 - X_i^2 \in M_{\mathbb{C}}$ for every i, we have

$$0 \leq L(p^*X_i^2p) \leq L(p^*p) \tag{8}$$

for all $p \in \mathbb{C}\langle \bar{X}\rangle$. Hence $L(p^*X_i^2p) = 0$ for all $p \in N$, i.e., $X_ip \in N$.

Because N is a left ideal, the map

$$\Lambda_i : \mathbb{C}\langle \bar{X}\rangle/N \to \mathbb{C}\langle \bar{X}\rangle/N,\ \overline{p} \mapsto \overline{X_ip}$$

is well-defined for each i. Obviously, it is linear and it is self-adjoint by the definition (7) of the scalar product. By (8), Λ_i is bounded with norm ≤ 1 and thus extends to a self-adjoint contraction $\hat{X}_i$ on E.

Let $\mathcal{F}$ denote the von Neumann subalgebra of $\mathcal{B}(E)$ generated by $\hat{X}_1, \ldots, \hat{X}_n$ and let τ denote the mapping

$$\sum_w a_w\hat{w} \mapsto \big\langle \sum_w a_w\hat{w}(1), 1\big\rangle = L\big(\sum_w a_w w\big). \tag{9}$$

τ is easily seen to be a tracial state on the algebra generated by $\hat{X}_1, \ldots, \hat{X}_n$. By continuity, τ extends uniquely to a faithful tracial state on $\mathcal{F}$. Moreover, 1 is a separating vector for τ. Hence $\mathcal{F}$ is a finite von Neumann algebra [Tak, Theorem V.2.4] and thus can be decomposed as $\mathcal{F} = \mathcal{F}_{\mathrm{I}} \oplus \mathcal{F}_{\mathrm{II}}$, where $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are finite von Neumann algebras of type I, respectively II [Tak, Theorem V.1.19]. Since L was an extremal tracial contraction state, we have $\mathcal{F}_{\mathrm{I}} = \{0\}$ or $\mathcal{F}_{\mathrm{II}} = \{0\}$. Assume that the latter holds. Then $\mathcal{F}$ is a finite type I von Neumann algebra, hence of type I_n for some $n \in \mathbb{N}$ and is isomorphic to $n \times n$ matrices over its center [Tak, Theorem V.1.27]. By (9), 1 is a trace vector for τ, so $n = 1$, i.e., $\mathcal{F}$ is abelian. Since E is separable, $\mathcal{F}$ can be written as a direct integral of I_1-factors (i.e., $\mathbb{C}$) [Tak, Theorem IV.8.21]. From this decomposition it follows by assumption (i) that $\tau(\hat{f}) \geq 0$. But $\tau(\hat{f}) = L(f) \notin \mathbb{R}_{\geq 0}$, contradiction.

Hence we may assume that $\mathcal{F}$ is a type II_1 von Neumann algebra with trace τ. As above, write $\mathcal{F}$ as a direct integral of II_1-factors and τ as a direct integral of (faithful) tracial states. It follows from assumption (i) that $\tau(\hat{f}) \geq 0$, again a contradiction to $\tau(\hat{f}) = L(f) \notin \mathbb{R}_{\geq 0}$. □

Lemma 3.13. *$M_{\mathbb{C}} \cap \mathbb{R}\langle \bar{X}\rangle = M_{\mathbb{R}}$. Moreover, if $f \in \mathbb{R}\langle \bar{X}\rangle$ is cyclically equivalent to an element of $M_{\mathbb{C}}$, then it is cyclically equivalent to an element of $M_{\mathbb{R}}$.*

Proof. Set $g_0 := 1$ and $g_i := 1 - X_i^2$ for $i \in \{1, \dots, n\}$ and suppose that

$$\sum_{i=0}^{n} \sum_{j} (p_{ij} + \mathbb{i} q_{ij})^* g_i (p_{ij} + \mathbb{i} q_{ij}) \in \mathbb{R}\langle \bar{X} \rangle$$

where $p_{ij}, q_{ij} \in \mathbb{R}\langle \bar{X} \rangle$. We have to show that this sum lies in $M_{\mathbb{R}}$. Since it lies in $\mathbb{R}\langle \bar{X} \rangle$, it is enough to show that it lies in $M_{\mathbb{R}}$ after adding its complex conjugate (which is the sum itself). But this is even true for each particular term in the sum since

$$(p_{ij} + \mathbb{i} q_{ij})^* g_i (p_{ij} + \mathbb{i} q_{ij}) + (p_{ij} - \mathbb{i} q_{ij})^* g_i (p_{ij} - \mathbb{i} q_{ij}) = 2(p_{ij}^* g_i p_{ij} + q_{ij}^* g_i q_{ij}) \in M_{\mathbb{R}}.$$

For the second statement, let $f + \sum_{i=1}^{t} [g_{i1}, g_{i2}] + \mathbb{i} \sum_{i=1}^{t} [h_{i1}, h_{i2}] \in M_{\mathbb{C}}$ for $g_{ij}, h_{ij} \in \mathbb{R}\langle \bar{X} \rangle$. By applying the complex conjugation and adding both equations, we obtain $f + \sum_{i=1}^{t} [g_{i1}, g_{i2}] \in M_{\mathbb{C}} \cap \mathbb{R}\langle \bar{X} \rangle = M_{\mathbb{R}}$. □

The polynomial from Remark 2.2 shows that the assumption $f = f^*$ cannot be omitted in the next two lemmas.

Lemma 3.14. *For $f \in \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle$, the following are equivalent:*

(i) *$\varphi(f) \geq 0$ for all tracial contraction states φ on $\mathbb{R}\langle \bar{X} \rangle$;*
(ii) *$\varphi(f) \geq 0$ for all tracial contraction states φ on $\mathbb{C}\langle \bar{X} \rangle$.*

Proof. If (ii) holds and $\varepsilon \in \mathbb{R}_{>0}$, then $f + \varepsilon$ is cyclically equivalent to an element of $M_{\mathbb{C}}$ by the implication (ii) $\Rightarrow$ (iii) in Theorem 3.12. Hence it is cyclically equivalent to an element of $M_{\mathbb{R}}$ by Lemma 3.13 and so $\varphi(f) \geq 0$ for all tracial contraction states φ on $\mathbb{R}\langle \bar{X} \rangle$ by Lemma 3.7. Conversely, suppose that (i) holds and let φ be a tracial contraction state on $\mathbb{C}\langle \bar{X} \rangle$. Then

$$\psi : \mathbb{R}\langle \bar{X} \rangle \to \mathbb{R}, \; p \mapsto \frac{\varphi(p) + \varphi(p)^*}{2}$$

is a tracial contraction state. Therefore $\varphi(f) = \psi(f) \geq 0$. □

Lemma 3.15. *For $f \in \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle$, the following are equivalent:*

(i) *$\operatorname{tr}(f(A_1, \dots, A_n)) \geq 0$ for all $s \in \mathbb{N}$ and self-adjoint $A_i \in \mathbb{R}^{s \times s}$;*
(ii) *$\operatorname{tr}(f(A_1, \dots, A_n)) \geq 0$ for all $s \in \mathbb{N}$ and self-adjoint $A_i \in \mathbb{C}^{s \times s}$.*

Proof. It is trivial that (ii) implies (i). For the other implication, we use the usual identification of a complex number $a + \mathbb{i} b$ $(a, b \in \mathbb{R})$ with the real matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Every self-adjoint complex matrix defines in this way a self-adjoint real matrix of double size with double trace. We leave the details to the reader. □

Corollary 3.16. *For $f \in \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle$, the following are equivalent:*

(i) *$\varphi(f) \geq 0$ for all tracial contraction states φ on $\mathbb{R}\langle \bar{X} \rangle$;*
(ii) *For every $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ is cyclically equivalent to an element of $M_{\mathbb{R}}$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 3.14, Theorem 3.12 and Lemma 3.13, while the converse follows from Lemma 3.7. □

The equivalence of (i), (ii) and (iv) in the next theorem is well-known [Had, R1, R2]. With condition (iv), one can reformulate Connes' Conjecture 1.1 without recourse to ultraproducts. Our contribution is the new condition (iii). The implications (i)⇒(ii)⇒(iii) are easy. The proof of (iii)⇒(iv) uses arguments similar to those of Hadwin [Had, p. 1789] and Rădulescu [R1, p. 232]. Since we work with polynomials, we can even argue in a simpler way and therefore include a proof. For the sake of completeness, we also include an elementary proof of (iv)⇒(i) which resembles the proof of [Con, Lemma 5.22].

Proposition 3.17. *For every separable* II_1*-factor* $\mathcal{F}$ *with trace* τ*, the following are equivalent:*

(i) *For every free ultrafilter* ω *on* $\mathbb{N}$*,* $\mathcal{F}$ *is embeddable in* $\mathcal{R}^\omega$*;*
(ii) *There is an ultrafilter* ω *on* $\mathbb{N}$ *such that* $\mathcal{F}$ *is embeddable in* $\mathcal{R}^\omega$*;*
(iii) *For each* $n \in \mathbb{N}$ *and* $f \in \mathbb{C}\langle \bar{X}\rangle$*, condition* (i) *from Conjecture* 1.5 *implies* $\tau(f(A_1,\dots,A_n)) \geq 0$ *for all self-adjoint contractions* $A_1,\dots,A_n \in \mathcal{F}$*;*
(iv) *For all* $\varepsilon \in \mathbb{R}_{>0}$*,* $n,k \in \mathbb{N}$ *and self-adjoint contractions* $A_1,\dots,A_n \in \mathcal{F}$*, there are* $s \in \mathbb{N}$ *and self-adjoint contractions* $B_1,\dots,B_n \in \mathbb{C}^{s\times s}$ *such that*
$$\left|\tau(w(A_1,\dots,A_n)) - \frac{1}{s}\operatorname{tr}(w(B_1,\dots,B_n))\right| < \varepsilon \qquad \text{for all } w \in \langle \bar{X}\rangle_k.$$

Proof. The implication (i)⇒(ii) is trivial.

For the proof of (ii)⇒(iii), let $f \in \mathbb{C}\langle \bar{X}\rangle$ satisfy condition (i) from Conjecture 1.5. Then $\tau_0(f(A_1,\dots,A_n)) \geq 0$ for all self-adjoint contractions $A_1,\dots,A_n \in \mathcal{R}$. Let ω be an ultrafilter on $\mathbb{N}$. By (ii), it suffices to show that $\tau_{0,\omega}(f(A_1,\dots,A_n)) \geq 0$ for all self-adjoint contractions $A_1,\dots,A_n \in \mathcal{R}^\omega$. By continuity, we may even assume that the A_i are not only contractions but there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\|A_i\| \leq 1-\varepsilon$. Then each A_i has a representative $(A_i^{(j)} + B_i^{(j)})_{j\in\mathbb{N}}$ such that each $A_i^{(j)}$ is a self-adjoint contraction in $\mathcal{R}$ and $(B_i^{(j)})_{j\in\mathbb{N}} \in I_\omega$. But then
$$\begin{aligned}\tau_{0,\omega}(f(A_1,\dots,A_n)) &= \lim_{j\to\omega} \tau_0(f(A_1^{(j)} + B_1^{(j)},\dots,A_n^{(j)} + B_n^{(j)}))\\ &= \lim_{j\to\omega} \tau_0(f(A_1^{(j)},\dots,A_n^{(j)})) \geq 0\end{aligned}$$
where the second equality follows from the fact that I_ω is an ideal and $\tau_{0,\omega}|_{I_\omega} = 0$.

To prove (iii)⇒(iv), let $\varepsilon > 0$ and $n,k \in \mathbb{N}$ be given. Consider the finite-dimensional $\mathbb{C}$-vector space $\mathbb{C}\langle \bar{X}\rangle_k$ and its dual space $\mathbb{C}\langle \bar{X}\rangle_k^\vee$. Let $C \subseteq \mathbb{C}\langle \bar{X}\rangle_k^\vee$ denote the closure of the convex hull of the set $T \subseteq \mathbb{C}\langle \bar{X}\rangle_k^\vee$ of all the linear forms
$$p \mapsto \frac{1}{s}\operatorname{tr}(p(\bar{B})) \quad (s \in \mathbb{N},\ \bar{B} \text{ an } n\text{-tuple of self-adjoint contractions in } \mathbb{C}^{s\times s}).$$
Now let an n-tuple $\bar{A}$ of self-adjoint contractions in $\mathcal{F}$ be given and consider $L \in \mathbb{C}\langle \bar{X}\rangle_k^\vee$ given by $L(p) = \tau(p(\bar{A}))$ for $p \in \mathbb{C}\langle \bar{X}\rangle_k$.

Assume $L \notin C$. By the complex Hahn-Banach separation theorem, we then find $f \in \mathbb{C}\langle \bar{X}\rangle_k \cong \mathbb{C}\langle \bar{X}\rangle_k^{\vee\vee}$ and $c \in \mathbb{R}$ such that $\operatorname{Re}(L(f)) < c < \operatorname{Re}(L'(f))$ for all $L' \in C$. Replacing f by $f - c$, we may assume $c = 0$. Then $L'(f + f^*) = L'(f) + L'(f)^* = 2\operatorname{Re}(L'(f)) > 0$ for all $L' \in C$ but $L(f + f^*) < 0$, contradicting (iii).

Therefore $L \in C$, i.e., every neighborhood of L in $\mathbb{C}\langle \bar{X}\rangle_k^\vee$ contains a convex combination of elements of T. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every such neighborhood also contains such a convex combination with rational coefficients. But building matrices in block diagonal form, it is easy to see that the set T is closed under such rational convex combinations.

To prove (iv)⇒(i), let $A_1, A_2, \dots$ be a sequence of self-adjoint contractions of $\mathcal{F}$ generating $\mathcal{F}$ as a von Neumann algebra. For each $k \in \mathbb{N}$, choose self-adjoint contractions $B_1^{(k)}, \dots, B_k^{(k)} \in \mathcal{R}$ satisfying

$$\left|\tau(w(A_1,\dots,A_k)) - \tau_{0,\omega}(w(B_1^{(k)},\dots,B_k^{(k)}))\right| < \frac{1}{k} \qquad \text{for each } w \in \langle X_1,\dots,X_k\rangle_k.$$

For each $i \in \mathbb{N}$, let $B_i \in \mathcal{R}^\omega$ be the self-adjoint contraction represented by the sequence $(B_i^{(k)})_{k\in\mathbb{N}}$ (with $B_i^{(k)} := 1$ for $i > k$). Then for all $n \in \mathbb{N}$ and $w \in \langle \bar{X}\rangle$ we have

$$\tau_{0,\omega}(w(B_1,\dots,B_n)) = \lim_{k\to\omega} \tau_0(w(B_1^{(k)},\dots,B_n^{(k)})) = \tau(w(A_1,\dots,A_n)). \tag{10}$$

There is a map ι that embeds the $*$-algebra generated by the A_i into $\mathcal{R}^\omega$ by mapping A_i to B_i for $i \in \mathbb{N}$. Indeed, if $A := \sum_w \lambda_w w(A_1,\dots,A_n) = 0$ and $B := \sum_w \lambda_w w(B_1,\dots,B_n)$, then (10) shows that $\|A\|_2 = \|B\|_2$. In particular, $\|A\|_2 = 0 \Leftrightarrow \|B\|_2 = 0$ which shows that ι is well-defined and injective. By (10), it is a trace-preserving $*$-homomorphism and therefore extends to an embedding $\iota : \mathcal{F} \hookrightarrow \mathcal{R}^\omega$. □

Theorem 3.18. *The following are equivalent:*

(i) *Connes' embedding conjecture* 1.1 *holds;*
(ii) *For* $\Bbbk = \mathbb{C}$*, conditions* (i) *from Conjecture* 1.5 *and the conditions from Theorem* 3.12 *are equivalent for all* $n \in \mathbb{N}$ *and* $f \in \mathbb{C}\langle \bar{X}\rangle$*;*
(iii) *For* $\Bbbk = \mathbb{R}$ *conditions* (i) *from Conjecture* 1.5 *and the conditions from Corollary* 3.16 *are equivalent for all* $n \in \mathbb{N}$ *and* $f \in \operatorname{Sym}\mathbb{R}\langle \bar{X}\rangle$.

Proof. First note that condition (i) from Conjecture 1.5 follows from the other conditions mentioned by Theorem 3.12 and Corollary 3.16. Now Proposition 3.17 shows that (i) and (ii) are equivalent. Finally, the equivalence of (ii) and (iii) follows from Proposition 2.3 together with Lemmas 3.14 and 3.15. □

Combining Theorem 3.18 with Theorem 3.12 and Corollary 3.16, we get the desired proof of Theorem 1.6.

4. Polynomials in two variables

In this section, we let $n = 2$ and write (X, Y) instead of (X_1, X_2). Moreover, we denote by $\pi : \mathbb{C}\langle X, Y\rangle \to \mathbb{C}[X, Y]$ the canonical ring epimorphism that lets the variables commute.

Definition 4.1. We call a word $w \in \langle X, Y\rangle$ *cyclically sorted* if it is cyclically equivalent to X^iY^j for some $i, j \in \mathbb{N}_0$. A polynomial $f \in \mathbb{C}\langle X, Y\rangle$ is called *cyclically sorted* if it is a linear combination of cyclically sorted words.

Proposition 4.2. *Let* $f \in \mathbb{C}\langle X, Y\rangle$ *be cyclically sorted. Suppose* $\pi(f) \geq 0$ *on* $[-1, 1]^2$*. Then* $f + \varepsilon$ *is cyclically equivalent to an element of* $M_\mathbb{R}$ *for all* $\varepsilon \in \mathbb{R}_{>0}$.

Proof. For each $g \in \mathbb{C}[X, Y]$, there is exactly one linear combination $\varrho(g)$ of words of the form X^iY^j $(i, j \in \mathbb{N}_0)$ such that $\pi(\varrho(g)) = g$. If $p, q \in \mathbb{C}\langle X, Y\rangle$ are cyclically sorted and satisfy $\pi(p) = \pi(q)$, then $p \overset{\text{cyc}}{\sim} q$. The hypothesis $\pi(f) \geq 0$ on $[-1, 1]^2$ implies $\pi(f) \in \mathbb{R}[X, Y]$ since the coefficients of f are essentially higher partial derivatives of f at the origin. Given $\varepsilon \in \mathbb{R}_{>0}$, it follows from Putinar's Theorem 1.7 that

$$\pi(f)+\varepsilon=\sum_i p_i^2+\sum_i q_i^2(1-X^2)+\sum_i r_i^2(1-Y^2)$$

for some $p_i, q_i, r_i \in \mathbb{R}[X,Y]$. This implies

$$f+\varepsilon \overset{\text{cyc}}{\sim} \sum_i \varrho(p_i)^*\varrho(p_i)+\sum_i \varrho(q_i)^*(1-X^2)\varrho(q_i)+\sum_i \varrho(r_i)(1-Y^2)\varrho(r_i)^* \in M_{\mathbb{R}}$$

because the expressions on both sides are cyclically sorted. $\square$

Example 4.3. Set

$$f:=(1-X^2)(1-Y^2)\in \mathbb{R}\langle X,Y\rangle.$$

Then $f+\varepsilon$ is cyclically equivalent to an element in $M_{\mathbb{R}}$ for every $\varepsilon \in \mathbb{R}_{>0}$. While this follows from Proposition 4.2, it can also be seen directly: We may assume $\varepsilon=\frac{1}{m}$ for some $m\in\mathbb{N}$ and note that

$$f+\frac{1}{m}\overset{\text{cyc}}{\sim}\Big(1-X^2+\frac{1}{m}X^{2m}\Big)(1-Y^2)+\frac{1}{m}(X^mY^2X^m+(1-X^{2m})).$$

The second term of this sum lies in $M_{\mathbb{R}}$ since

$$1-X^{2m}=\sum_{k=0}^{m-1}X^k(1-X^2)X^k,$$

and we use Remark 3.6 to see that the first term is cyclically equivalent to

$$\frac{1}{m}(1-Y^2)+\frac{1}{m}(1-X^2)\Big(\sum_{k=0}^{m-2}(m-1-k)X^k(1-Y^2)X^k\Big)(1-X^2)\in M_{\mathbb{R}}.$$

For $\varepsilon=0$, $f+\varepsilon$ is not cyclically equivalent to an element of $M_{\mathbb{R}}$. In fact, it is an easy exercise to show that $\pi(f)\notin\pi(M_{\mathbb{R}})$.

Example 4.4. The polynomial

$$f:=YX^4Y+XY^4X-3XY^2X+1\in \operatorname{Sym}\mathbb{R}\langle X,Y\rangle$$

is a noncommutative cyclically sorted version of the Motzkin polynomial $\pi(f)$. The Motzkin polynomial is probably the most well-known example of a polynomial which is nonnegative on $\mathbb{R}^2$ but not a sum of squares of polynomials [Rez]. By Proposition 4.2, $f+\varepsilon$ is for each $\varepsilon\in\mathbb{R}_{>0}$ cyclically equivalent to an element of $M_{\mathbb{R}}$. This shows in particular that $\operatorname{tr}(f(A,B))\geq 0$ for all $s\in\mathbb{N}$ and all self-adjoint contractions $A,B\in\mathbb{C}^{s\times s}$. Since $\pi(f)\geq 0$ on (any square in) $\mathbb{R}^2$, we can use the same reasoning together with a scaling argument to see that $\operatorname{tr}(f(A,B))\geq 0$ for all $s\in\mathbb{N}$ and all self-adjoint matrices $A,B\in\mathbb{C}^{s\times s}$, a fact for which we do not know a direct proof. However, a direct proof that $f+\varepsilon$ is for all $\varepsilon\in\mathbb{R}_{>0}$ cyclically equivalent to an element of $M_{\mathbb{R}}$ can be obtained as in the previous example since

$$f\overset{\text{cyc}}{\sim}Y(1-X^2)^2Y+X(1-Y^2)^2X+(1-X^2)(1-Y^2).$$

Note that $f(A,B)$ is not positive semidefinite for all self-adjoint contractions $A,B\in\mathbb{R}^{2\times 2}$, since for

$$A:=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\qquad\text{and}\qquad B:=\begin{pmatrix}-1&0\\0&1\end{pmatrix},$$

$$f(A,B)=\frac{1}{2}\begin{pmatrix}1&-3\\-3&1\end{pmatrix}$$

is clearly not positive semidefinite.

5. Bounds

In this section, we use valuation theory [P-C], basic first order logic and model theory of real closed fields [Pre] to derive certain bounds for Conjecture 1.5. For the moment, let (i) and (ii) refer to the respective conditions for $\Bbbk = \mathbb{R}$ in Conjecture 1.5. As we have seen in Theorem 1.6, Connes conjecture is equivalent to the implication (i)⇒(ii) for $f \in \operatorname{Sym}\mathbb{R}\langle\bar{X}\rangle$. Here we show that this implication must actually hold in a stronger form if it holds at all. Suppose that Connes' conjecture holds and we are given $f \in \operatorname{Sym}\mathbb{R}\langle\bar{X}\rangle$ and $\varepsilon \in \mathbb{R}_{>0}$. Then there are two bounds. First, there is a bound on the size of the matrices on which the nonnegativity condition in (i) has to be tested. Second, there is a bound on the degree complexity of the representation of $f + \varepsilon$ (for this particular ε) in (ii). These bounds depend only on ε, the number of variables, the degree of f and the size of the coefficients of f (rather than on f itself). Moreover, the bounds are computable from this data (in the sense of recursion theory). Unfortunately, the rather nonconstructive methods yielding these bounds do not allow for further specification of the kind of dependence. We will first prove a certain technical version of Corollary 3.16 which is valid not only over $\mathbb{R}$ but over any real closed field (see Proposition 5.7).

Let us recall some facts from the theory of ordered fields. Suppose R is a real closed field. Let $\leq$ denote the ordering of R and

$$\mathcal{O} := \{a \in R \mid |a| \leq N \text{ for some } N \in \mathbb{N}\}$$

the convex hull of $\mathbb{Z}$ in R. This is a valuation ring with (unique) maximal ideal $\mathfrak{m}$ given by

$$\mathfrak{m} = \{a \in R \mid N|a| \leq 1 \text{ for all } N \in \mathbb{N}\}.$$

The residue field $\mathcal{O}/\mathfrak{m}$ is again a real closed field (cf. [Pre, 8.6] or [P-C, II §4 Lemma 17]), but this time archimedean and thus embeds uniquely into $\mathbb{R}$ [P-C, II §3 Satz 3]. We therefore always assume $\mathcal{O}/\mathfrak{m} \subseteq \mathbb{R}$. Moreover, we find at least one embedding $\varrho : \mathcal{O}/\mathfrak{m} \hookrightarrow \mathcal{O} \subseteq R$ such that $\overline{\varrho(x)} = x$ for all $x \in \mathcal{O}/\mathfrak{m}$ [P-C, III §2 Satz 6]. We extend the canonical homomorphism $\mathcal{O} \to \mathcal{O}/\mathfrak{m} \subseteq \mathbb{R}$ to a ring homomorphism

$$\mathcal{O}\langle\bar{X}\rangle \to \mathbb{R}\langle\bar{X}\rangle,\ f \mapsto \overline{f}$$

sending X_i to X_i. Similarly, ϱ can be extended to polynomials.

The quadratic module $M_R \subseteq \operatorname{Sym} R\langle\bar{X}\rangle$ generated by $1-X_1^2, \dots, 1-X_n^2$ consists exactly of the sums of elements of the form

$$g^*g \qquad \text{and} \qquad g^*(1-X_i^2)g \qquad (1 \leq i \leq n,\ g \in R\langle\bar{X}\rangle). \tag{11}$$

Now consider only elements of this form of degree at most $2k$ ($k \in \mathbb{N}$) and call the set of all sums of such elements $M_{R,k}$. Then $M_{R,k}$ is a convex cone in the R-vector space $\operatorname{Sym} R\langle\bar{X}\rangle_{2k}$ which is (perhaps strictly) contained in $M_R \cap R\langle\bar{X}\rangle_{2k}$. Clearly, $M_R = \bigcup_{k\in\mathbb{N}} M_{R,k}$.

Since we will no longer be concerned with complex matrices but with matrices over real closed fields, it seems more appropriate to speak of *symmetric* matrices rather than self-adjoint ones.

Proposition 5.1. *Suppose $k \in \mathbb{N}$. Let U denote the subspace of $\operatorname{Sym}\mathbb{R}\langle\bar{X}\rangle_{2k}$ of those elements which are cyclically equivalent to 0. Then $M_{\mathbb{R},k} + U$ is closed in $\operatorname{Sym}\mathbb{R}\langle\bar{X}\rangle_{2k}$.*

Proof. Let $\pi : \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle_{2k} \to (\operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle_{2k})/U =: V$ be the canonical projection. Then $M_{\mathbb{R},k} + U = \pi^{-1}(\pi(M_{\mathbb{R},k}))$. Hence, it suffices to show that the convex cone $\pi(M_{\mathbb{R},k})$ is closed in V. By Carathéodory's theorem (see e.g. [Hol, p. 40, Exercise 1.8]), each element of $\pi(M_{\mathbb{R},k})$ can be written as the image of a sum of at most m terms of the form (11) where $m := \dim V$. Setting $p_0 := 1$ and $p_i := 1 - X_i^2$ for $i \in \{1, \dots, n\}$, we see that $\pi(M_{\mathbb{R},k})$ is the image of the map

$$\Phi : \begin{cases} \mathbb{R}\langle \bar{X} \rangle_k^m \times \mathbb{R}\langle \bar{X} \rangle_{k-1}^m \times \cdots \times \mathbb{R}\langle \bar{X} \rangle_{k-1}^m \to V \\ (g_{01}, \dots, g_{0m}, \quad \dots \quad , g_{n1}, \dots, g_{nm}) \mapsto \pi\left(\sum_{i=0}^n \sum_{j=1}^m g_{ij}^* p_i g_{ij}\right). \end{cases}$$

We claim that $\Phi^{-1}(0) = \{0\}$. To show this, suppose

$$h := \sum_{i=0}^{n} \sum_{j=1}^{m} g_{ij}^* p_i g_{ij} \overset{\text{cyc}}{\sim} 0. \tag{12}$$

Let $s \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathbb{R}^{s \times s}$ be symmetric with $\|A_i\| < 1$. Then $1 - A_i^2$ is a positive definite and can be written as $1 - A_i^2 = B_i^2$ for some symmetric invertible $B_i \in \mathbb{R}^{s \times s}$. It is convenient to let B_0 denote the identity matrix in $\mathbb{R}^{s \times s}$. Denoting by e_t the t-th unit vector of $\mathbb{R}^s$, it follows from (12) that

$$\sum_{t=1}^{s} \sum_{i=0}^{n} \sum_{j=1}^{m} \langle B_i g_{ij}(\bar{A}) e_t, B_i g_{ij}(\bar{A}) e_t \rangle = \operatorname{tr}(h(A_1, \dots, A_n)) = 0.$$

Consequently, we get $B_i g_{ij}(\bar{A}) e_t = 0$ and hence $g_{ij}(\bar{A}) e_t = 0$ for all i, j, t. This shows that $g_{ij}(A_1, \dots, A_n) = 0$ for all symmetric $A_i \in \mathbb{R}^{s \times s}$ with $\|A_i\| < 1$. By continuity, the same holds for all symmetric contractions $A_i \in \mathbb{R}^{s \times s}$. Hence [KS, Proposition 2.3] implies that $g_{ij} = 0$. This shows that $\Phi^{-1}(0) = \{0\}$. Together with the fact that Φ is homogeneous, [PS, Lemma 2.7] shows that Φ is a proper and therefore a closed map. In particular, its image $\pi(M_{\mathbb{R},k})$ is closed in V. □

In the following lemma, we will apply *Tarski's transfer principle*, i.e., the fact that exactly the same first order sentences with symbols 0, 1, +, ·, ≤ hold in each real closed field [Pre, 5.3].

Lemma 5.2. *Let $k \in \mathbb{N}$ and U be the subspace of $\operatorname{Sym} R\langle \bar{X} \rangle_{2k}$ of those elements which are cyclically equivalent to 0. Suppose that $f \in \operatorname{Sym} R\langle \bar{X} \rangle_{2k} \setminus (M_{R,k} + U)$. Then there is a linear map $L : \operatorname{Sym} R\langle \bar{X} \rangle_{2k} \to R$ such that $L(M_{R,k}) \subseteq R_{\geq 0}$, $L|_U = 0$, $L(1) = 1$ and $L(f) < 0$.*

Proof. We first prove this for $R = \mathbb{R}$. Consider the convex cone $M_{\mathbb{R},k} + U$ in $\operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle_{2k}$ which is closed by Proposition 5.1. Separating this cone from the cone spanned by a little ball around f (use e.g. [Hol, p. 15, §4B Corollary]), we find a linear map $L_0 : \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle_{2k} \to \mathbb{R}$ such that $L_0(M_{\mathbb{R},k} + U) \subseteq \mathbb{R}_{\geq 0}$ and $L_0(f) < 0$. Since $1 \in M_{\mathbb{R},k}$, we have $L_0(1) \geq 0$. If $L_0(1) > 0$, then $L := \frac{L_0}{L_0(1)}$ has the desired properties. If $L_0(1) = 0$, then we set $L := L_1 + \lambda L_0$ where

$$L_1 : \operatorname{Sym} \mathbb{R}\langle \bar{X} \rangle_{2k} \to \mathbb{R}, \quad g \mapsto g(0)$$

and $\lambda \in \mathbb{R}_{>0}$ is sufficiently large to ensure that $L(f) < 0$. This proves the statement for $R = \mathbb{R}$.

The general case follows by Tarski's transfer principle once we know that the statement can *for fixed* $k, n \in \mathbb{N}$ be expressed in the first order language with symbols 0, 1, +, ·, ≤. But this is indeed possible: To model $f \in \operatorname{Sym} R\langle \bar{X} \rangle_{2k}$, use universal quantifiers for the finitely many coefficients that a polynomial of degree $2k$ in n variables can have. The condition $f \notin M_{R,k} + U$ can also be written down in this language by using Carathéodory's theorem as in the proof of Proposition 5.1. The existence of the linear map L can be expressed by existential quantifiers for the values of L on a basis of $\operatorname{Sym} R\langle \bar{X}\rangle_{2k}$. $\square$

By Lemma 3.11 and (5), we find for every word $w \in \langle \bar{X}\rangle$ an $N_w \in \mathbb{N}$ such that $N_w \pm (w + w^*) \in M_{\mathbb{Q}}$. Moreover, we find for each $k \in \mathbb{N}$ some $d_k \geq k$ such that

$$2N_w \pm (w + w^*) \in M_{\mathbb{Q},d_k} \subseteq M_{R,d_k} \qquad \text{for all } w \in \langle \bar{X}\rangle_{2k}. \tag{13}$$

Lemma 5.3. *Suppose $k \in \mathbb{N}$ and $f \in \operatorname{Sym} R\langle \bar{X}\rangle_{2k}$ is not cyclically equivalent to an element of $M_{R,k}$. Then there is a linear map $L : R\langle \bar{X}\rangle_{2k} \to R$ such that $L(f) < 0$,*

(a) *$L(pq) = L(qp)$ for all $p, q \in R\langle \bar{X}\rangle$ such that $pq \in R\langle \bar{X}\rangle_{2k}$;*
(b) *$L(M_{R,k}) \subseteq R_{\geq 0}$;*
(c) *$|L(w)| \leq N_w$ for all $w \in \langle \bar{X}\rangle_{2k}$;*
(d) *$L(1) = 1$;*
(e) *$L(p^*) = L(p)$ for all $p \in R\langle \bar{X}\rangle_{2k}$.*

Proof. Set $d := d_k \geq k$. By Lemma 5.2, we find a linear map $L_0 : \operatorname{Sym} R\langle \bar{X}\rangle_{2d} \to R$ such that $L_0(M_{R,d}) \subseteq R_{\geq 0}$, $L_0|_U = 0$, $L_0(1) = 1$ and $L_0(f) < 0$ where $U \subseteq \operatorname{Sym} R\langle \bar{X}\rangle_{2d}$ is the subspace of polynomials that are cyclically equivalent to 0. The linear map

$$L : R\langle \bar{X}\rangle_{2k} \to R, \quad p \mapsto L_0\left(\frac{p + p^*}{2}\right)$$

extends the restriction of L_0 to $\operatorname{Sym} R\langle \bar{X}\rangle_{2k}$ which shows (b),(d) and $L(f) < 0$. Property (e) is clear from the definition of L. By (13), we have

$$2(N_w \pm L(w)) = 2N_w L(1) \pm (L(w) + L(w^*)) = L_0(2N_w \pm (w + w^*)) \geq 0$$

which yields (c). To show (a), suppose $p, q \in R\langle \bar{X}\rangle$ are such that $pq \in R\langle \bar{X}\rangle_{2k}$. Then $pq \overset{\text{cyc}}{\sim} qp$ and $(pq)^* \overset{\text{cyc}}{\sim} (qp)^*$ imply that $pq + (pq)^* \overset{\text{cyc}}{\sim} qp + (qp)^*$. This shows $pq + (pq)^* - (qp + (qp)^*) \in U$ whence $2L(pq) = L_0(pq + (pq)^*) = L_0(qp + (qp)^*) = 2L(qp)$. $\square$

Lemma 5.4. *Suppose $k \in \mathbb{N}$ and $f \in \operatorname{Sym} \mathcal{O}\langle \bar{X}\rangle_{2k}$ is not cyclically equivalent to an element of $M_{R,k}$. Then there is a linear map $L : (\mathcal{O}/\mathfrak{m})\langle \bar{X}\rangle_{2k} \to \mathcal{O}/\mathfrak{m}$ that satisfies $L(\overline{f}) \leq 0$ and conditions* (a)–(e) *from Lemma* 5.3 (*with R replaced by $\mathcal{O}/\mathfrak{m}$*).

Proof. Let L_0 be one of the linear maps whose existence has been shown in the previous lemma. Property (c) (with L replaced by L_0) implies that $L_0(\mathcal{O}\langle \bar{X}\rangle) \subseteq \mathcal{O}$. We can thus define the map

$$L : (\mathcal{O}/\mathfrak{m})\langle \bar{X}\rangle_{2k} \to \mathcal{O}/\mathfrak{m}, \quad p \mapsto \overline{L_0(\varrho(p))}.$$

Using that $\overline{\varrho(\lambda)} = \lambda$ for all $\lambda \in \mathcal{O}/\mathfrak{m}$, we see that L is $\mathcal{O}/\mathfrak{m}$-linear. We know that $\varrho(\overline{f}) - f$ has all its coefficients in $\mathfrak{m}$. Because of property (c), this shows that $L_0(\varrho(\overline{f}) - f) \in \mathfrak{m}$ whence

$$L(\overline{f}) = \overline{L_0(\varrho(\overline{f}))} = \overline{L_0(f)} + \overline{L_0(\varrho(\overline{f}) - f)} = \overline{L_0(f)} \leq 0.$$

Moreover, it is easy to see that L inherits properties (a)–(e) from L_0. $\square$

Lemma 5.5. *Suppose $k \in \mathbb{N}$ and $f \in \operatorname{Sym} \mathcal{O}\langle \bar{X} \rangle_{2k}$ is not cyclically equivalent to an element of $M_{R,k}$. Then there is a linear map $L : \mathbb{R}\langle \bar{X} \rangle_{2k} \to \mathbb{R}$ that satisfies $L(\overline{f}) \leq 0$ and conditions* (a)–(e) *from Lemma* 5.3 (*with R replaced by $\mathbb{R}$*).

Proof. Let L_0 be one of the linear maps whose existence has been shown in the previous lemma. Let x_w and y_w be variables in the formal language of first order logic where w ranges over all $w \in \langle \bar{X} \rangle_{2k}$. Build up a formula Φ with free variables x_w and y_w in the first order language with symbols 0, 1, $+$, $\cdot$, $\leq$ expressing that (over the real closed field R where the formula is interpreted) $L(\sum_w y_w w) \leq 0$ and conditions (a)–(e) from Lemma 5.3 hold for the linear map $L : R\langle \bar{X} \rangle_{2k} \to R$ given by $L(w) = x_w$. Compare the second part of the proof of Lemma 5.2 for some details on how this can be done. By Lemma 5.4, Φ holds in the real closed field $\mathcal{O}/\mathfrak{m}$ when x_w is interpreted as $L_0(w)$ and y_w is interpreted as the coefficient of w in $\overline{f}$. Define another formula Ψ with free variables y_w which arises from Φ by quantifying all x_w existentially. Then Ψ holds in $\mathcal{O}/\mathfrak{m}$ when the y_w are interpreted as the coefficients of f. By the substructure completeness of the theory of real closed fields [Pre, 5.1,4.7], Ψ holds also in the real closed extension field $\mathbb{R}$ of $\mathcal{O}/\mathfrak{m}$ under the same interpretation of the y_w. □

Lemma 5.6. *Suppose $f \in \operatorname{Sym} R\langle \bar{X} \rangle_{2k}$ has all its coefficients in $\mathfrak{m}$. Then for each $\varepsilon \in R_{>0} \setminus \mathfrak{m}$, we have $f + \varepsilon \in M_{R,d_k}$.*

Proof. Without loss of generality, we may assume that $f = a(w + w^*)$ with $a \in \mathfrak{m}$ and $w \in \langle \bar{X} \rangle_{2k}$. Then

$$\begin{aligned} f + \varepsilon &= a(w + w^*) + |a|N_w + (\varepsilon - |a|N_w) \\ &= |a|(N_w + \operatorname{sign}(a)(w + w^*)) + (\varepsilon - |a|N_w) \in M_{R,d_k} \end{aligned}$$

since $\varepsilon - |a|N_w \geq 0$ and $N_w \pm (w + w^*) \in M_{\mathbb{Q},d_k} \subseteq M_{R,d_k}$ by (13). □

Proposition 5.7. *Suppose $f \in \operatorname{Sym} \mathcal{O}\langle \bar{X} \rangle$ and $\varphi(\overline{f}) \geq 0$ for all tracial contraction states φ on $\mathbb{R}\langle \bar{X} \rangle$. Then for all $\varepsilon \in R_{>0} \setminus \mathfrak{m}$, $f + \varepsilon$ is cyclically equivalent to an element of M_R.*

Proof. We show the contraposition, i.e., we assume that we have $N \in \mathbb{N}$ such that $f + \frac{1}{N}$ is *not* cyclically equivalent to an element of M_R and find a tracial contraction state φ on $\mathbb{R}\langle \bar{X} \rangle$ such that $\varphi(\overline{f}) < 0$. Let (a)–(e) refer to the conditions from Lemma 5.3 with R replaced by $\mathbb{R}$. Lemma 5.5 provides us for each $k \in \mathbb{N}$ such that $2k \geq \deg f$ with a linear map $L_k : \mathbb{R}\langle \bar{X} \rangle_{2k} \to \mathbb{R}$ satisfying $L_k(\overline{f} + \frac{1}{N}) \leq 0$ and (a)–(e). To each L_k, we associate a point P_k in the product space $S := \prod_{w \in \langle \bar{X} \rangle} [-N_w, N_w]$ by setting $P_k(w) := L_k(w)$ if $w \in \langle \bar{X} \rangle_{2k}$ and $P_k(w) := 0$ if $w \in \langle \bar{X} \rangle \setminus \langle \bar{X} \rangle_{2k}$. Since S is compact by Tychonoff's theorem, the sequence $(P_k)_k$ has a subsequence converging to some $P \in S$. Define the linear map $\varphi : \mathbb{R}\langle \bar{X} \rangle \to \mathbb{R}$ by $\varphi(w) := P(w)$ for all $w \in \langle \bar{X} \rangle$. Using (b), (d) together with $M_R = \bigcup_{k \in \mathbb{N}} M_{R,k}$, (a), (e) and Lemma 3.7, it is easy to see that φ is a tracial contraction state such that $\varphi(\overline{f} + \frac{1}{N}) \leq 0$ and therefore $\varphi(\overline{f}) \leq -\frac{1}{N} < 0$. □

Theorem 5.8. *Suppose that Connes' embedding conjecture* 1.1 *holds. Then there is a computable function $N : \mathbb{N} \to \mathbb{N}$ such that for all $t \in \mathbb{N}$ the following is true: Whenever $n \in \mathbb{N}$ with $n \leq t$, $f \in \operatorname{Sym} \mathbb{R}\langle X_1, \dots, X_n \rangle$ is of degree $\leq t$, has absolute value of its coefficients bounded by t and satisfies* $\operatorname{tr}(f(A_1, \dots, A_n)) \geq 0$ *for all* *symmetric contractions* $A_i \in \mathbb{R}^{N(t)\times N(t)}$*, then* $f+\frac{1}{t}$ *is cyclically equivalent to an element of* $M_{\mathbb{R},N(t)}$.

Proof. For technical reasons, it is convenient to replace the condition $n \le t$ in the statement by the condition $n = t$. This does not affect the generality of the theorem since

$$M^{(t)}_{\mathbb{R},N} \cap \mathbb{R}\langle \bar{X}\rangle = M^{(n)}_{\mathbb{R},N} \qquad \text{for } n \le t.$$

Most facts about finite-dimensional real Euclidean vector spaces carry over from $\mathbb{R}$ to any real closed field by Tarski's transfer principle. We will therefore use concepts like symmetric contractions over the real closed fields R and $\mathcal{O}/\mathfrak{m}$. For a matrix $A \in \mathcal{O}^{s\times s}$, we can apply the map $\mathcal{O} \to \mathcal{O}/\mathfrak{m}$ entrywise and get a matrix $\overline{A} \in (\mathcal{O}/\mathfrak{m})^{s\times s}$. For every symmetric contraction $A \in (\mathcal{O}/\mathfrak{m})^{s\times s}$, there is a symmetric contraction $B \in R^{s\times s}$ with all its entries in $\mathcal{O}$ such that $\overline{B} = A$.

Claim 1. For fixed $t \in \mathbb{N}$, the following infinitely many conditions (a), (b), (c), (d$_s$) and (e$_s$) ($s \in \mathbb{N}$) cannot be satisfied simultaneously.

(a) R is a real closed field;
(b) $f \in \operatorname{Sym} R\langle X_1,\dots,X_t\rangle$ is of degree at most t;
(c) The absolute value of the coefficients of f is bounded by t;
(d$_s$) $\operatorname{tr}(f(A_1,\dots,A_t)) \ge 0$ for all symmetric contractions $A_i \in R^{s\times s}$;
(e$_s$) $f+\frac{1}{t}$ is not cyclically equivalent to an element of $M^{(t)}_{R,s}$.

Proof of Claim 1. Assuming these conditions, we obtain the following.

(c′) $f \in \mathcal{O}\langle X_1,\dots,X_t\rangle$;
(b′) $\overline{f} \in \operatorname{Sym}\mathbb{R}\langle X_1,\dots,X_t\rangle$;
(d′$_s$) $\operatorname{tr}(\overline{f}(A_1,\dots,A_t)) \ge 0$ for all symmetric contractions $A_i \in \mathbb{R}^{s\times s}$;
(e′) $\overline{f}+\frac{1}{2t}$ is not cyclically equivalent to an element of $M^{(t)}_{\mathbb{R}}$.

Of course, (c′) follows from (c) by the definition of $\mathcal{O}$. Because of (c′), we can consider $\overline{f}$ and from (b) it is clear that (b′) holds. It is easy to see that (d$_s$) implies (d′$_s$) for all symmetric contractions $A_i \in (\mathcal{O}/\mathfrak{m})^{s\times s}$. With Tarski's transfer principle and the fact that $\mathcal{O}/\mathfrak{m}$ and $\mathbb{R}$ are real closed, it is easy to extend this from $\mathcal{O}/\mathfrak{m}$ to $\mathbb{R}$ (cf. Lemma 5.5). Now assume that (e') does not hold, i.e., $\overline{f}+\frac{1}{2t}$ is cyclically equivalent to an element of $M^{(t)}_{\mathbb{R},s}$ for some $s \in \mathbb{N}$. By Tarski's principle (use again Carathéodory's theorem to express this in first order logic), we get

$$\varrho(\overline{f}) + \frac{1}{2t} \text{ is cyclically equivalent to an element of } M^{(t)}_{R,s} \subseteq M^{(t)}_{R,d_s}. \tag{14}$$

From the fact that $f-\varrho(\overline{f})$ has all its coefficients in $\mathfrak{m}$ and Lemma 5.6, it follows that $f-\varrho(\overline{f})+\frac{1}{2t} \in M^{(k)}_{R,d_s}$. Combining this with (14) yields that $f+\frac{1}{t}$ is cyclically equivalent to an element of $M^{(k)}_{R,d_s}$ which contradicts (e_{d_s}). Finally, use Proposition 5.7 to see that (b′), (d′$_s$) ($s \in \mathbb{N}$) and (e′) cannot be satisfied simultaneously if the algebraic version 1.5 of Connes' conjecture holds. But this algebraic version is equivalent to Connes' conjecture by Theorem 1.6. This proves Claim 1.

As we have just seen, a lot of specifications (like the degree in (b), the concrete bound for the absolute value of the coefficients in (c), etc.) are not needed for Claim 1 but they ensure that the next claim holds.

Claim 2. For fixed $t \in \mathbb{N}$, the above conditions (a), (b), (c), (d$_s$) and (e$_s$) ($s \in \mathbb{N}$) can be expressed in the language of first order logic with symbols 0, 1, $+$, $\cdot$, $\le$ and new constants for the finitely many coefficients that a polynomial

$f \in R\langle X_1, \dots, X_t\rangle$ of degree at most t can have. Moreover, there is a *decidable* (i.e., recursive) set of formulas in this language corresponding to (a), (b), (c), (d_s) and (e_s).

Proof of Claim 2. Concerning (a), write down the axioms for real closed fields. For (b), we have introduced the new constants. The natural number t in (c) can be written as $1 + \cdots + 1$. There are several good ways to express (d_s) by a formula for each fixed s. Finally, use Carathéodory's theorem once more to translate (e_s) into such a formula for each fixed s.

THE ALGORITHM. We describe a procedure how to calculate the function N that we are looking for. The program takes $t \in \mathbb{N}$ and yields a suitable $N(t)$. Let the program generate successively all words of length $1, 2, 3, \dots$ over the finite alphabet of the language from Claim 1. Every time a word has been generated, let the program check whether this is by chance a formal proof of $0 = 1$ in the first order predicate calculus that uses only axioms from the set of formulas from Claim 1 (this can be checked since this set is decidable by Claim 2). When the program encounters such a formal proof, let it terminate after outputting the smallest number $N(t)$ such that the found formal proof uses as axioms only (a), (b), (c), (d_s) and (e_s) for $s \leq N(t)$.

Proof of termination. Since the set of allowed axioms is inconsistent by Claim 1, $0 = 1$ is a logical consequence of it. By Gödel's completeness theorem, the algorithm will thus eventually terminate.

Proof of correctness. The number $N(t)$ has the desired properties because $\mathbb{R}$ is real closed and conditions (a), (b), (c), (d_s), (e_s) for $s := N(t)$ must be inconsistent (observe that (d_{k+1}) implies (d_k) and (e_{k+1}) implies (e_k) for all $k \in \mathbb{N}$). $\square$

Note that the information that the bound $N(t)$ is computable from t means that it can in a certain sense not grow "too" fast when $t \to \infty$. By a diagonal argument, it is indeed easy to see that there are functions $\mathbb{N} \to \mathbb{N}$ growing faster than any computable function. On the other hand, the described algorithm computing $N(t)$ from t has a tremendous complexity and is therefore purely theoretical. If one is not interested in the information that N is computable, one can replace Gödel's completeness theorem by the compactness theorem from first order logic.

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Igor Klep, Univerza v Ljubljani, Oddelek za matematiko Inštituta za matematiko, fiziko in mehaniko, Jadranska 19, 1111 Ljubljana, Slovénie

E-mail address: igor.klep@fmf.uni-lj.si

Markus Schweighofer, Universität Konstanz, Fachbereich Mathematik und Statistik, 78457 Konstanz, Allemagne

E-mail address: markus.schweighofer@uni-konstanz.de