Sums of squares, moments and optimization

Markus Schweighofer

Institut de Recherche Mathématique de Rennes

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A system of inequalities

might get easier to solve then you add a few other inequalities

to it.

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$$-X^{12}+938X^{9}-56629X^{6}-54758X^{10}+109984X^{7}-55694X^{4}-110449X^{8}+219494X^{5}-109513X^{2}+468X^{11}+110448X^{3}+468X-54756\geq 0$$

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- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

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is a set of polynomials which are

for obvious reasons ≥ 0 on $S = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$.

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is the preorder generated by g.

We call $P \subseteq \mathbb{R}[X]$ a preorder if $\mathbb{R}[X]^2 \subseteq P$, $P + P \subseteq P$ and $PP \subseteq P$.

Everything works for basic closed semialgebraic sets

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- Technical problems disappear.

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Thomas Jacobi, Alexander Prestel and Mihai Putinar.

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- The converse is trivial.
- Rediscovered by Stengle. Usually attributed to him.

Gilbert Stengle: A Nullstellensatz and a Positivstellensatz in semialgebraic geometry
Math. Ann. **207**, 87–97 (1974)

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Tentative proof.

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Tentative proof. Suppose $\forall q \in T : qf \notin 1 + T$, i.e., $-1 \notin T - Tf$.

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(Like in the proof of the intermediate value theorem.)

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Hope:

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Then, for any $p \in P$, $p(x) \in p(X) + I = p + I \subseteq P + P \subseteq P$.

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Then, for any $p \in P$, $p(x) \in p(X) + I = p + I \subseteq P + P \subseteq P$.
Hence $p(x) \in P \cap \mathbb{R}$

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Then, for any $p \in P$, $p(x) \in p(X) + I = p + I \subseteq P + P \subseteq P$.
Hence $p(x) \in P \cap \mathbb{R} = [0, \infty)$, i.e., $p(x) \geq 0$.

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Then, for any $p \in P$, $p(x) \in p(X) + I = p + I \subseteq P + P \subseteq P$.
Hence $p(x) \in P \cap \mathbb{R} = [0, \infty)$, i.e., $p(x) \geq 0$.
In particular, $g(x) \geq 0$ and $-f(x) \geq 0$.

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$$\forall p, q \in \mathbb{R}[X] : (-pq \in P \implies p \in P \text{ or } q \in P)$$

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If $-pq \in P$, $p \notin P$ and $q \notin P$, then $-1 \in P + Pp$ and $-1 \in P + Pq$,

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If $-pq \in P$, $p \notin P$ and $q \notin P$, then $-1 \in P + Pp$ and $-1 \in P + Pq$, i.e., there are $a, b, c, d \in P$ such that

$$\begin{aligned} -1 &= a + bp \\ -1 &= c + dq \end{aligned}$$

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$$\forall p, q \in \mathbb{R}[X] : (-pq \in P \implies p \in P \text{ or } q \in P)$$

If $-pq \in P$, $p \notin P$ and $q \notin P$, then $-1 \in P + Pp$ and $-1 \in P + Pq$, i.e., there are $a, b, c, d \in P$ such that

$$\left.\begin{aligned} -1 = a + bp &\implies a + 1 = -bp \\ -1 = c + dq &\implies c + 1 = -dq \end{aligned}\right\} \implies -1 = ac + a + c - bdpq \in P.$$

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$$\mathbb{R} \to \mathbb{R}[X]/I \subseteq \mathsf{qf}(\mathbb{R}[X]/I) =: K.$$

The ordering $\leq$ on K is defined via P such that

$$\forall p \in \mathbb{R}[X] : (p + I \geq 0 \iff p \in P).$$

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Have $y \in K^n$. Want $x \in \mathbb{R}^n$!

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Artin and Schreier: Every ordered field K can be extended to a real closed field.

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Remarks about the proof

- Distinction between good and bad case is not necessary.
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- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
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- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

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Konrad Schmüdgen: The K-moment problem for compact semi–algebraic sets
Math. Ann. **289**, No. 2, 203–206 (1991)

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- Denominatorfree version of following formulation of the Positivstellensatz:

$$f > 0 \text{ on } S \iff \exists \varepsilon > 0 : \exists q \in T : qf \in \varepsilon + T$$

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The set $\{p \in \mathbb{R}[X] \mid \exists N \in \mathbb{N} : N \pm p \in T\}$ contains $\mathbb{R}$ and is closed under addition. Because of the two equalities

$$NN' + pp' = \frac{1}{2}((N+p)(N'+p') + (N-p)(N'-p'))$$

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$$\frac{M+1}{2} \pm X_i = \frac{1}{2}\left((X_i \pm 1)^2 + \left(M - \sum_{j=1}^{n} X_j^2\right) + \sum_{j \neq i} X_j^2\right).$$

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Marshall Stone, Donald Dubois, Richard Kadison, Eberhard Becker

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George Pólya: Über positive Darstellung von Polynomen
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Jesús De Loera, Francisco Santos: An effective version of Pólya's theorem on positive definite forms
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But $f_\varepsilon := \sum_{|\alpha|=d} a_\alpha (X_1)_\varepsilon^{\alpha_1} \cdots (X_n)_\varepsilon^{\alpha_n} \to f$ uniformly on Δ for $\varepsilon \to 0$.

Notation for the whole week (recapitulation)

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g \in \mathbb{R}[X]$ the polynomial defining. . .
- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

Remember that S is assumed to be compact.

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$$\text{Weak version} \overset{\text{Pólya}}{\Longrightarrow} \text{Strong version}$$

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- X_i might not be in T.

Proof.

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Therefore $F > 0$ on $[0, \infty)^{n+1} \setminus \{0\}$. By Pólya's Theorem, there is $k \in \mathbb{N}$ such that $(X_1 + \dots + X_n + Z)^k F$ has no negative coefficients. Finally, substitute $N - (X_1 + \dots + X_n)$ for Z.

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- Main idea was introduction of new coordinates lying in T and summing up to a natural number N (barycentric coordinates).
- The fact that these coordinates sum up to N allowed rewriting f as a homogeneous polynomial in the new coordinates and made the denominator from Pólya's Theorem harmless.

- Rewriting did not change the values of f on

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This is bad since possible nonpositivity on Δ is kept.

- Therefore we were forced to establish positivity on S in advance by the lemma. But the lemma behaves bad with respect to degree complexity.

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- Need other coordinates satisfying more algebraic relations. In addition, S must inside Δ be defined by an equation since any rewrite step which lets f invariant on S, lets f invariant on the Zariski–closure of S.

- Idea: Try to take g itself as an additional coordinate.

Proof. Suppose $f > 0$ on S. To show: $f \in T$. By weak version, w.l.o.g. $X_1, \dots, X_n \in T$ and we find $N \in \mathbb{N}$ such that $N - (X_1 + \cdots + X_n + g) \in T$.

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$$V := \{(x_1, \dots, x_n, y, z) \in \Delta \mid y = g(x)\} \subseteq \Delta$$

Finally, substitute g for Y and $N - (X_1 + \dots + X_n + g)$ for Z.

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Then $f > 0$ on V.

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Then $f > 0$ on V. For big $\lambda \in \mathbb{R}$, $h := f + \lambda(Y - g)^2 \in \mathbb{R}[X, Y, Z]$ is positive on Δ.

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Proof. Suppose $f > 0$ on S. To show: $f \in T$. By weak version, w.l.o.g. $X_1, \dots, X_n \in T$ and we find $N \in \mathbb{N}$ such that $N - (X_1 + \dots + X_n + g) \in T$.

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- The proof is again an effective construction, in particular, it avoids the Positivstellensatz.

- Degree of $h := f + \lambda(Y - g)^2$ depends only on $\deg f$ and $\deg g$ and not on geometric properties of f.

First proof:
Optimization of polynomials on compact semialgebraic sets
preprint

Second proof:
An algorithmic approach to Schmüdgen's Positivstellensatz
Journal of Pure and Applied Algebra **166**, 307–319 (2002)

Consequences of second proof:
On the complexity of Schmüdgen's Positivstellensatz
to appear in Journal of Complexity

Notation for the whole week (recapitulation)

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g \in \mathbb{R}[X]$ the polynomial defining. . .
- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

Remember that S is assumed to be compact.

The S–moment problem

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The S–moment problem

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To be more precise, denote by $\mathcal{M}^1(A)$ the set of all probability measures on a subset A of $\mathbb{R}^n$. Then the question is:

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To be more precise, denote by $\mathcal{M}^1(A)$ the set of all probability measures on a subset A of $\mathbb{R}^n$. Then the question is:

For which real families $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ is it true that

$$\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : a_\alpha = \int X^\alpha d\mu$$

holds?

Schmüdgen's solution to the moment problem. Write $g = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha$, $c_\alpha \in \mathbb{R}$. For every real family $(a_\alpha)_{\alpha \in \mathbb{N}^n}$ are equivalent:

(1) $a_0 = 1$ and for all real families $(b_\alpha)_{\alpha \in \mathbb{N}^n}$ with finite support,

$$\sum_{\alpha,\beta \in \mathbb{N}^n} b_\alpha b_\beta a_{\alpha+\beta} \geq 0 \qquad \text{and} \qquad \sum_{\alpha,\beta,\gamma \in \mathbb{N}^n} b_\alpha b_\beta c_\gamma a_{\alpha+\beta+\gamma} \geq 0.$$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : a_\alpha = \int X^\alpha d\mu$

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(1) $L(1) = 1$ and for all real families $(b_\alpha)_{\alpha\in\mathbb{N}^n}$ with finite support,

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(2) $\exists \mu \in \mathcal{M}^1(S) : \forall \alpha \in \mathbb{N}^n : L(X^\alpha) = \int X^\alpha d\mu$

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- Take a dual standpoint:

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

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Schmüdgen's solution $\Downarrow$ to the moment problem

$$\boxed{f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T) \subseteq [0, \infty)\}}$$

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Schmüdgen's $\Downarrow$ Positivstellensatz

$$\boxed{f^* = \sup\{a \in \mathbb{R} \mid f - a \in T\}}$$

Introduce finite–dimensional approximations $T_k \subseteq \mathbb{R}[X]_k$ of $T \subseteq \mathbb{R}[X]$.

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$$T_k := \sum \mathbb{R}[X]_d{}^2 + \sum \mathbb{R}[X]_e{}^2\, g \qquad \text{convex cone}$$

$$= \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2, \deg \sigma \leq k, \deg(\tau g) \leq k\right\}$$

for arbitrary $k \in \mathcal{N} := \{s \in \mathbb{N} \mid s \geq \max\{\deg g, \deg f\}\}$.

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Warning: Never confuse T_k with $T \cap \mathbb{R}[X]_k \supseteq T_k$.

We saw that

$f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T) \subseteq [0, \infty)\}$ and

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In analogy to this, we set

$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(T_k) \subseteq [0, \infty)\}$ and
$D_k^* = \sup\{a \in \mathbb{R} \mid f - a \in T_k\}$

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for every $k \in \mathcal{N}$.

$P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$ are the optimal values of the following pair of optimization problems. . .

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
$f - a \in T_k$

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$.

Proof.

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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Proof. $P_k^* \leq f^*$ because $p \mapsto p(x)$ feasible for (P_k) for $x \in S$.

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$D_k^* \leq P_k^*$: $L(f) - a = L(f) - aL(1) = L(f - a) \subseteq L(T_k) \subseteq [0, \infty)$

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Clear: P_k^* and D_k^* increase.

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Clear: P_k^* and D_k^* increase. $\lim_{k \to \infty} D_k^* \to f^*$: If $a < f^*$, then $f - a \in T_k$ for some $k \in \mathcal{N}$ by Schmüdgen's Positivstellensatz.

$$
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Clear: P_k^* and D_k^* increase. $\lim_{k \to \infty} D_k^* \to f^*$: If $a < f^*$, then $f - a \in T_k$ for some $k \in \mathcal{N}$ by Schmüdgen's Positivstellensatz. Then a is feasible for (D_k) whence $a \leq D_k^*$.

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\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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\end{array}
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$$
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(D_k) & \text{maximize} & a & \text{subject to} & a \in \mathbb{R} \text{ and} \\
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Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all $k \in \mathcal{N}$.

Proof. $P_k^* \leq f^*$ because $p \mapsto p(x)$ feasible for (P_k) for $x \in S$.

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(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
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Theorem. There exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on f such that

$$
f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.
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On the complexity of Schmüdgen's Positivstellensatz
to appear in Journal of Complexity

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
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Dependance on f can be made explicit. Proof hints to make dependance on g explicit for concrete g.

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Dependance on f can be made explicit. Proof hints to make dependance on g explicit for concrete g. Main idea: Second approach to Schmüdgen's Positivstellensatz via Pólya.

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f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.
$$

In practice: Convergence usually very fast,
often $D_k^* = P_k^* = f^*$ for small k.

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

Schmüdgen's Positivstellensatz implies convergence of D_k^* and therefore of P_k^*.

What can we know from Schmüdgen's solution to the moment problem?

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A priori nothing!

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Schmüdgen's Positivstellensatz implies convergence of D_k^* and therefore of P_k^*.

What can we know from Schmüdgen's solution to the moment problem?
A priori nothing! But with additional compactness arguments involving Tychonoff's Theorem, the following. . .

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

Theorem. Suppose that L_k solves (P_k) nearly to optimality ($k \in \mathcal{N}$).

$$\forall d \in \mathbb{N} : \forall \varepsilon > 0 : \exists k_0 \in \mathcal{N} \cap [d, \infty) : \forall k \geq k_0 : \exists \mu \in \mathcal{M}^1(S^*) :$$
$$\left\| \left(L_k(X^\alpha) - \int X^\alpha d\mu \right)_{|\alpha| \leq d} \right\| < \varepsilon.$$

Optimization of polynomials on compact semialgebraic sets
preprint

$$(P_k) \quad \text{minimize} \quad L(f) \quad \text{subject to} \quad \begin{array}{l} L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\ L(1) = 1 \text{ and} \\ L(T_k) \subseteq [0, \infty) \end{array}$$

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In particular, if $S^* = \{x^*\}$ is a singleton,

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In particular, if $S^* = \{x^*\}$ is a singleton, then

$$\lim_{k \to \infty} (L_k(X_1), \ldots, L_k(X_n)) = x^*.$$

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$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
$f - a \in T_k$

Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

- "Strong duality"

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

- "Strong duality"
- "Weak duality" $D_k^* \leq P_k^*$ always holds.

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- First proved by using duality theory from semidefinite programming

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Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

- "Strong duality"
- "Weak duality" $D_k^* \leq P_k^*$ always holds.
- First proved by using duality theory from semidefinite programming since (P_k) and (D_k) can be translated into semidefinite programs and are as such dual to each other.

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

Sketch of Marshall's direct proof.

(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear, $L(1) = 1$ and $L(T_k) \subseteq [0, \infty)$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and $f - a \in T_k$

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Sketch of Marshall's direct proof. It suffices to show that T_k is closed in $\mathbb{R}[X]_k$.

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$$(p_1, \dots, p_s, q_1, \dots, q_s) \mapsto \sum_{i=1}^{s} p_i^2 + \sum_{i=1}^{s} q_i^2 g.$$

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$$(p_1, \ldots, p_s, q_1, \ldots, q_s) \mapsto \sum_{i=1}^{s} p_i^2 + \sum_{i=1}^{s} q_i^2 g.$$

This map is quadratically homogeneous and injective.

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Notation for the whole week (recapitulation)

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g \in \mathbb{R}[X]$ the polynomial defining. . .
- . . . the set $S := \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$. . .
- . . . and the preorder $T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 g$

Remember that S is assumed to be compact.

Optimization

We consider the problem of minimizing f on S. So we want to compute numerically the infimum (minimum if $S \neq \emptyset$)

$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Introduce finite–dimensional approximations $T_k \subseteq \mathbb{R}[X]_k$ of $T \subseteq \mathbb{R}[X]$.

$$\mathbb{R}[X]_k := \{p \mid p \in \mathbb{R}[X], \deg p \leq k\} \qquad \text{real vector space}$$

$$T_k := \sum \mathbb{R}[X]_d{}^2 + \sum \mathbb{R}[X]_e{}^2\, g \qquad \text{convex cone}$$

$$= \left\{\sigma + \tau g \mid \sigma, \tau \in \sum \mathbb{R}[X]^2, \deg \sigma \leq k, \deg(\tau g) \leq k\right\}$$

for arbitrary $k \in \mathcal{N} := \{s \in \mathbb{N} \mid s \geq \max\{\deg g, \deg f\}\}$.

Here $d := \max\{m \in \mathbb{N} \mid 2m \leq k\}$ and $e := \max\{m \in \mathbb{N} \mid 2m+\deg g \leq k\}$.

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(P_k) & \text{minimize} & L(f) & \text{subject to} \quad L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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\end{array}
$$

Denote the optimal values of these optimization problems by $P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$, respectively.

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\begin{array}{lllll}
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If S has nonempty interior, then $D_k^* = P_k^*$.

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$$\lim_{k\to\infty} (L_k(X_1), \ldots, L_k(X_n)) = x^*.$$

- (P_k) and (D_k) can be translated in a primal–dual pair of semidefinite programs which can be solved efficiently (Lasserre).

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- Feasible solutions of the semidefinite program corresponding to (D_k) give rise to a lower bound a of f^* together with a certificate (advantage) in form of a representation of $f - a$ proving $f - a \in T_k$.

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- Method converges from below to the infimum (advantage in many applications).

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- Method converges from below to the infimum (advantage in many applications).

- Method converges to unique minimizers. Disadvantage: Possibly from outside the set.

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Additional instruments for detecting optimality and extracting solutions

- If L is an optimal solution of (P_k), $x := (L(X_1), \ldots, L(X_n)) \in S$ and $L(f) = f(x)$, then $L(f) = P_k^* \leq f^* \leq f(x) = L(f)$

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Raul Curto, Lawrence Fialkow: The truncated complex K–moment problem Trans. Am. Math. Soc. **352**, No. 6, 2825–2855 (2000)

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How to solve the relaxations?

(P_k) minimize $L(f)$ subject to $L : \mathbb{R}[X]_k \to \mathbb{R}$ is linear,
$L(1) = 1$ and
$L(T_k) \subseteq [0, \infty)$

(D_k) maximize a subject to $a \in \mathbb{R}$ and
$f - a \in T_k$

- Optimization of a linear function on a convex set.

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- The cone $S\mathbb{R}_+^{s \times s}$ of positive semidefinite symmetric matrices has such a barrier function:

$$X \mapsto -\ln \det X$$

- Semidefinite programming is an extension of linear programming.

- Linear programming: Optimization of a linear function $\mathbb{R}^s \longrightarrow \mathbb{R}$ on the intersection of the selfdual cone $[0, \infty)^s$ with an affine subspace of $\mathbb{R}^s$.

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- A lot of efficient semidefinite programming solvers are freely available.

Sums of squares and semidefinite matrices

Let v a column vector of length s whose entries generate the vector space $\mathbb{R}[X]_d$. Then $\sum \mathbb{R}[X]_d^2 = \left\{ v^T G v \mid G \in S\mathbb{R}_+^{s \times s} \right\}$.

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$$\sum_{i=1}^{t} p_i^2 = (Av)^T Av = v^T (\underbrace{A^T A}_{\in S\mathbb{R}_+^{s \times s}}) v.$$

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Shows also what we used for showing strong duality.

Sums of squares and semidefinite matrices

Remember that, for $k \in \mathcal{N}$,

$$T_k = \sum \mathbb{R}[X]_d{}^2 + \sum \mathbb{R}[X]_e{}^2 g$$

where $d := \max\{m \in \mathbb{N} \mid 2m \le k\}$ and
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Let v a column vector of length s whose entries generate the vector space $\mathbb{R}[X]_d$. Let w a column vector of length t whose entries generate the vector space $\mathbb{R}[X]_e$. Then

$$T_k = \{v^T G v + w^T H w g \mid G \in S\mathbb{R}^{s \times s}, H \in S\mathbb{R}^{t \times t}\}.$$

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With little elaborations this gives the translation of (P_k) into a semidefinite program.

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We just outlined how (P_k) can be formulated as a semidefinite program. For (P_k) this is even easier. To express that a linear map $L : \mathbb{R}[X]_k \to \mathbb{R}$ satisfies $L(T_k) \subset [0, \infty)$, one writes down that the matrices representing the following bilinear forms are positive semidefinite:

$$\mathbb{R}[X]_d \times \mathbb{R}[X]_d \to \mathbb{R} : (p, q) \mapsto L(pq) \qquad \text{and}$$
$$\mathbb{R}[X]_e \times \mathbb{R}[X]_e \to \mathbb{R} : (p, q) \mapsto L(pqg)$$

Implementations

- Henrion and Lasserre: GloptiPoly
  `http://www.laas.fr/~henrion/software/gloptipoly/`

- Prajna, Papachristodoulou, Parrilo: SOSTOOLS
  `http://control.ee.ethz.ch/~parrilo/sostools/`

- Both use the free SeDuMi solver by Jos Sturm

- But they need MATLAB and the MATLAB Symbolic Toolbox

Example: The maximum cut problem

Given a graph, i.e., an $n \in \mathbb{N}$ (number of nodes) and a set

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$$\begin{aligned} &\text{maximize} && \sum_{(i,j)\in E} \frac{1}{2}(1 - x_i x_j) \\ &\text{subject to} && x_i^2 = 1 \text{ for all } i \in \{1,\ldots,n\} \end{aligned}$$

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