POP23 - Future Trends in Polynomial Optimization

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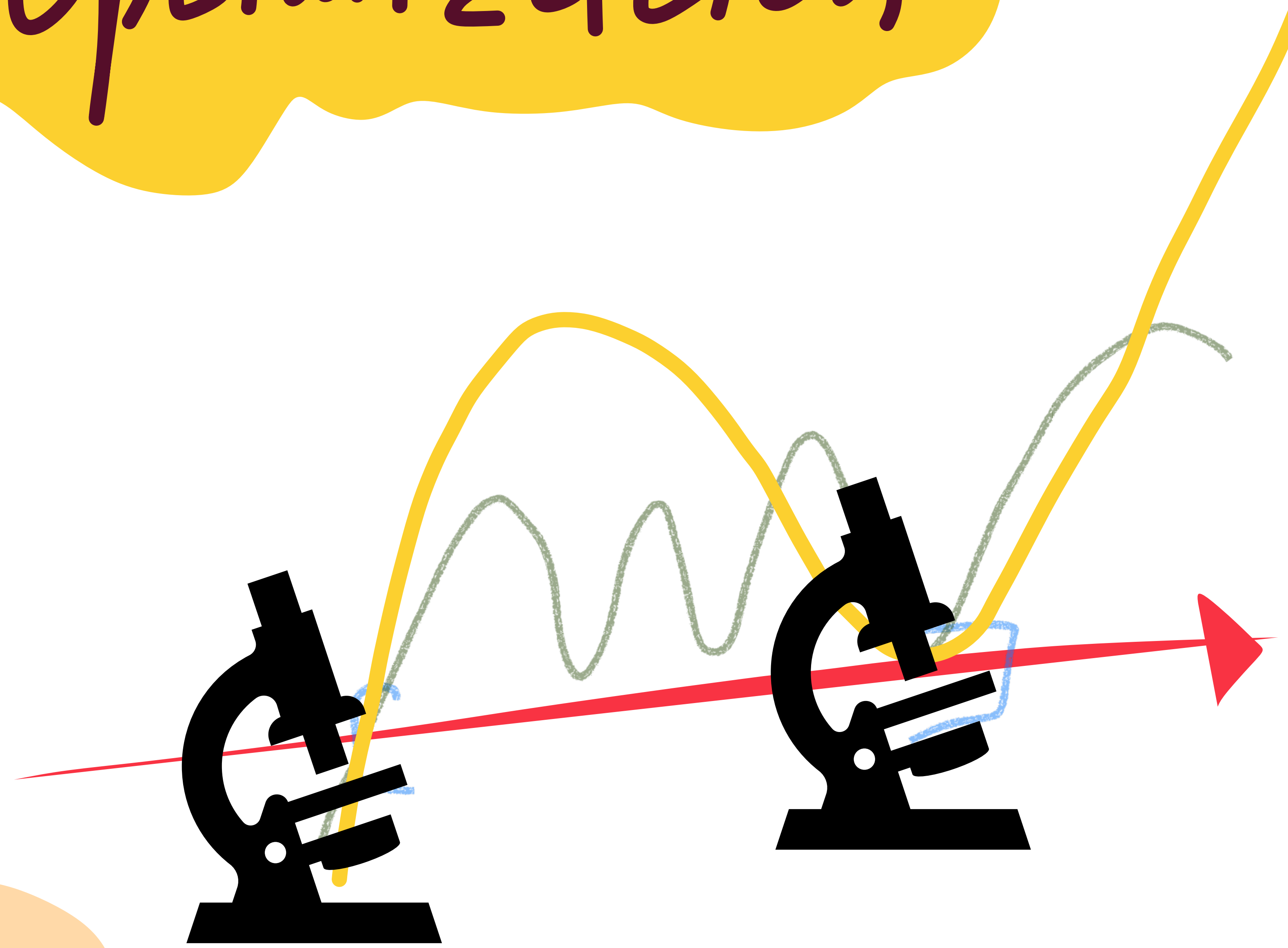
LAAS-CNRS, Toulouse



Pure states for polynomial nonnegativity certificates in the presence of zeros

Markus Schweighofer (Universität Konstanz)

(joint work with Luis Felipe Vargas)

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variables

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A quadratic module M of $\mathbb{R}[x]$ is called Archimedean if $M + \mathbb{Z} = \mathbb{R}[x]$.

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$$S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\} \quad \text{sphere}$$

$$B^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \quad \text{ball}$$

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$M_{B^n} := \sum \mathbb{R}[x]^2 + \sum \mathbb{R}[x]^2\left(1 - \sum_{i=1}^{n} x_i^2\right)$ is an Archimedean quadratic module with $S(M_{B^n}) = B^n$.

Theorem 4 (Putinar, 1993) Let M be an Archimedean quadratic module and $p \in \mathbb{R}[x]$. Then

$$p > 0 \text{ on } S(M) \implies p \in M.$$

What if $p \geq 0$ on $S(M)$?

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Example 5 $p := (x_1^2 + \dots + x_5^2)^2 - 4(x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_4^2 + x_2^2x_5^2 + x_3^2x_5^2)$
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$p(x_1, 0, x_3, x_4, 0) = (x_1^2 + x_3^2 + x_4^2)^2 - 4(x_1^2x_3^2 + x_1^2x_4^2) = (x_1^2 - x_3^2 - x_4^2)^2$

p has infinitely many zeros on S^4.

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Nie 2014

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I don't tell you what is a pure state since we have introduced test states.

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Here we will deal with the case of infinitely many zeros!

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Definition 6 Let I be an ideal and M be a quadratic module of $\mathbb{R}[x]$.
Let $v \in I$ and $a \in \mathbb{R}^n$. We call $\varphi : I \to \mathbb{R}$ a **test state** on I for M at a wrt. v if

- $\varphi(v) = 1$,
- $\varphi(I \cap M) \subseteq \mathbb{R}_{\geq 0}$,
- $\forall p, q \in I : \varphi(p+q) = \varphi(p) + \varphi(q)$ and
- $\forall p \in \mathbb{R}[x] : \forall q \in I : \varphi(pq) = p(a)\,\varphi(q)$.

https://arxiv.org/abs/2310.12853

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Example 7

$n = 1$, $\mathbb{R}[x] = \mathbb{R}[x_1]$

$I = \mathbb{R}[x]$

$M = M_{B^1}$

$a = 0$

$u = 1$

If $\varphi : I \to \mathbb{R}$ is a test state for M at a wrt. u, then $\varphi(p) = \varphi(pu) = p(0)\,\varphi(u) = p(0)$ for all $p \in \mathbb{R}[x]$.
Conversely, this defines a test state.
Evaluation at zero is the only test state!

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Derivative at zero is the only test state!

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Example 9

n arbitrary

$$I = \sum_{i,j=1}^{n} \mathbb{R}[x] x_i x_j$$

$$M = M_{B^n}$$

$$a = 0$$

$$u = x_1^2 + \ldots + x_n^2$$

If $\varphi : I \to \mathbb{R}$ is a test state, then $A := (\varphi(x_i x_j))_{1 \leq i,j \leq n}$ is psd and $\varphi(p) = \operatorname{tr}((\nabla^2 f(0))A)$ for all $p \in \mathbb{R}[x]$. Up to a positive constant, the test states are exactly the non-zero conic combinations of second directional derivatives at 0.

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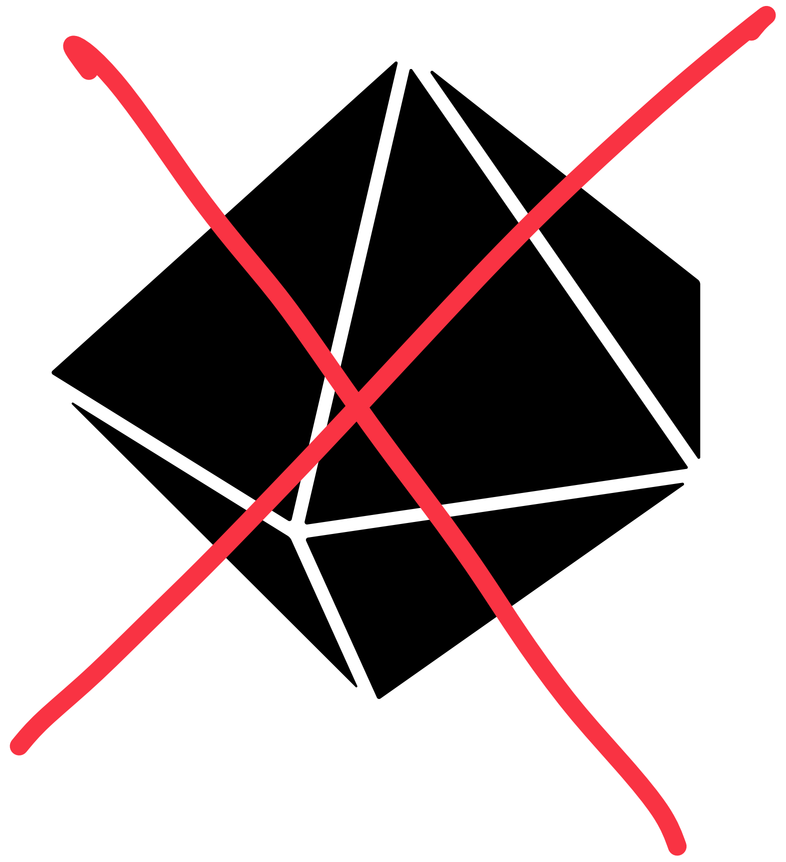

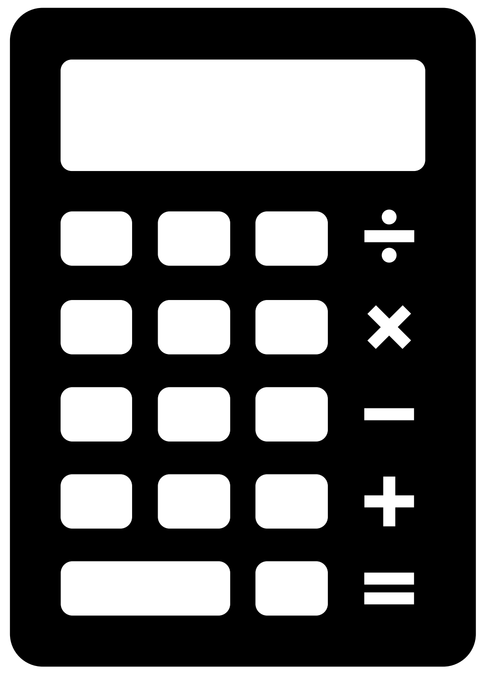

Warning 10

We do not think that test states always have a nice geometric interpretation. Although they are associated with a point $a \in \mathbb{R}^n$, we think that they are of algebraic nature in general.

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In particular, $f \in M$.

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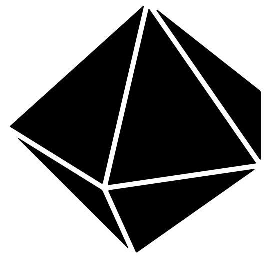
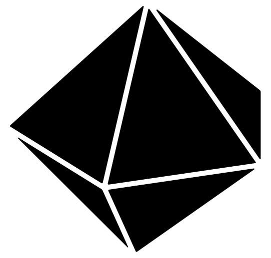
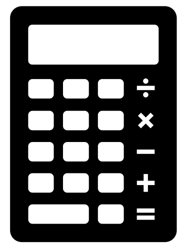
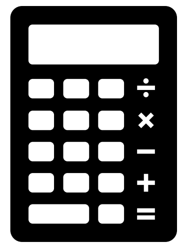
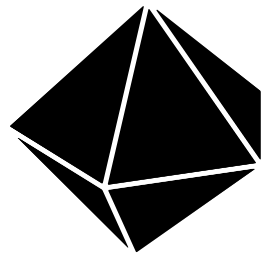
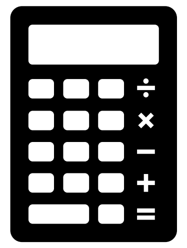
(a) $f \geq 0$ on $S(M)$ 







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Theorem 11

Let $F \subseteq \mathbb{R}[x]$ generate the ideal I.

Let M be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \implies u(a) = 0)$

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It collapses to ~~Putinar's Theorem!~~

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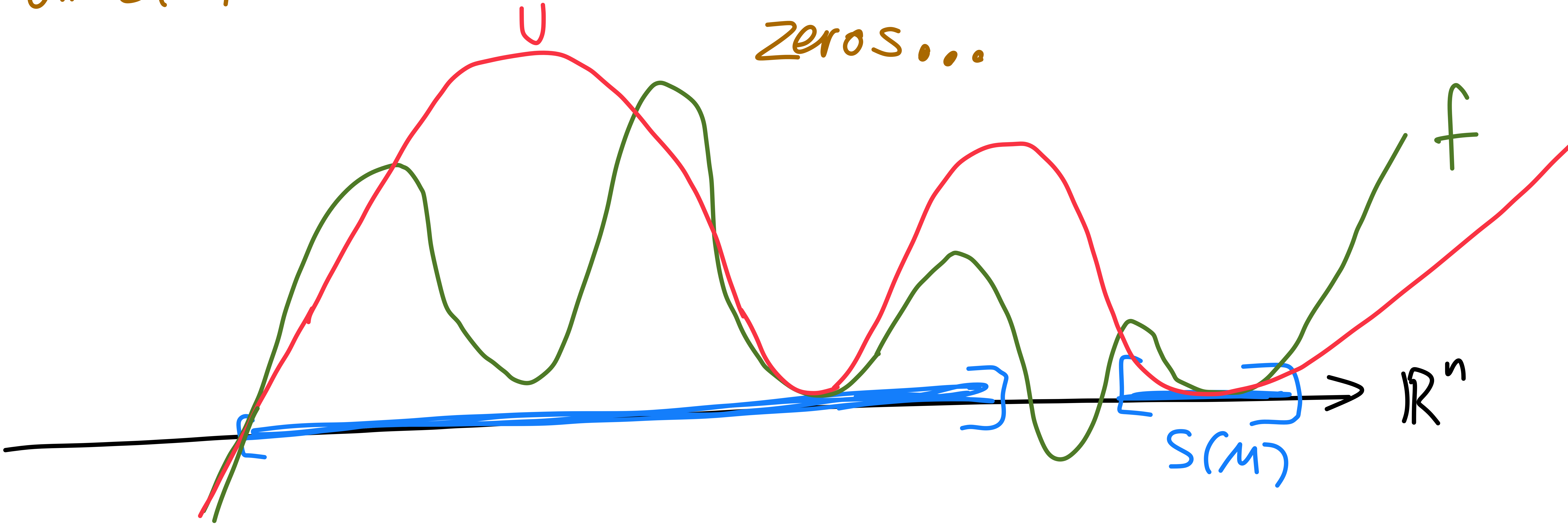
Under these hypotheses, one can show that actually, even more:

$\exists \varepsilon > 0$: $u - \varepsilon f,\ f - \varepsilon u \in M$ and hence ≥ 0 on $S(M)$.

$$\exists \varepsilon > 0 : \big(f - \varepsilon u \geq 0 \text{ on } S(M) \\ \& \ u - \varepsilon f \geq 0 \text{ on } S(M) \big)$$

means that not only that f and u have the same zeros on S(M) but also that they behave similarly near these zeros...



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Step-by-step strategy

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Then, there is $\varepsilon > 0$ such that $f - \varepsilon v \in M$. In particular, $f \in M$.

Let M be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f \in \mathbb{R}[x]$ with $f \geq 0$ on $S(M)$. Want to prove $f \in M$.

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Step 3. Prove (e) by using geometric arguments or algebraic identities inside the ideal I or both.

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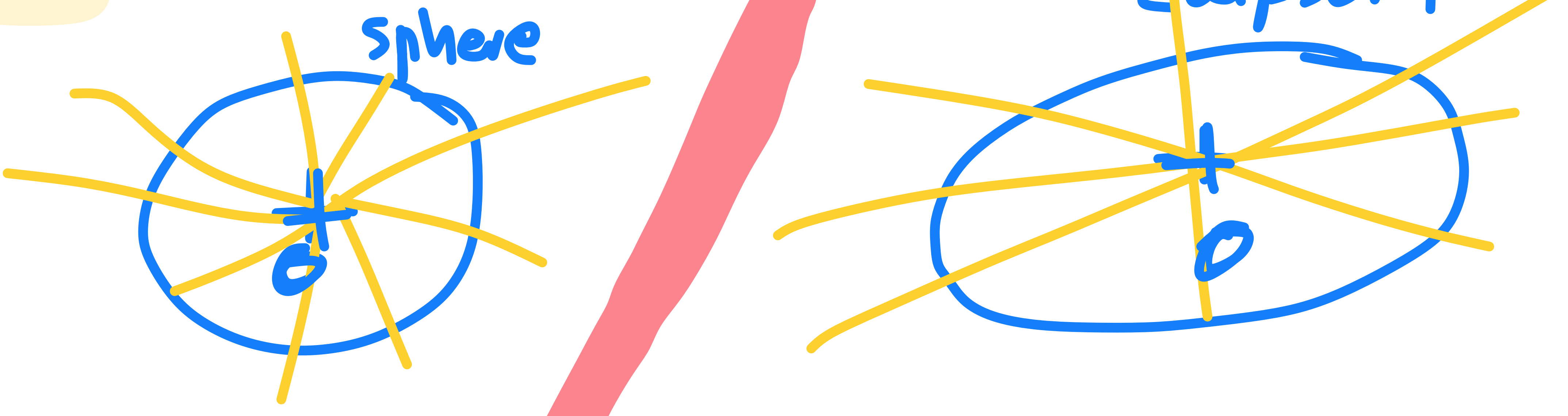

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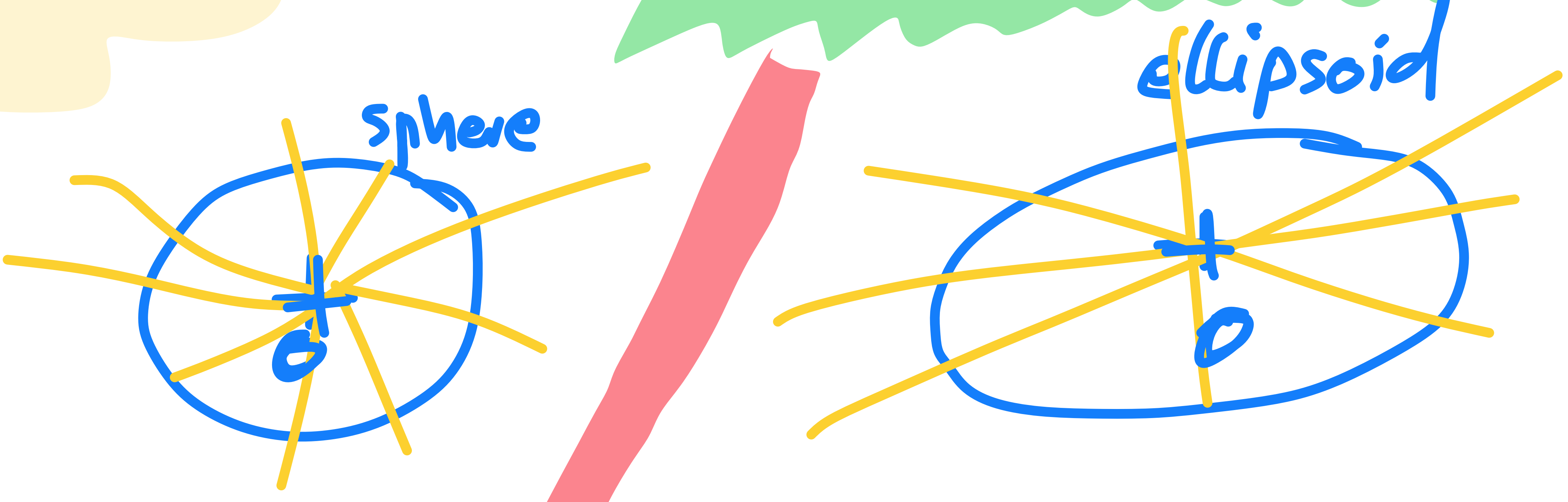

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Step 3. Let φ be a test state on $I := \mathbb{R}[x]f$ for M at a zero a of f on the ellipsoid. Then

$$1 = \varphi(u) = \left(\sum_{i=1}^{5} a_i^2\right)\varphi(f).$$ Hence $\varphi(f) > 0$. $\square$

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Theorem 13 (Reznick, 1995) Let $f \in \mathbb{R}[x]$ be a pd form. Then there is $r \in \mathbb{N}_0$ such that $(x_1^2 + \ldots + x_n^2)^r f \in \sum \mathbb{R}[x]^2$

Proposition 14 (de Klerk, Laurent, Parrilo 2005)

For every form $p \in \mathbb{R}[x]$ of even degree,

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... implies together with Thm. 4 (Putinar) Reznick's Thm. 13 but ...

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Hildebrand classified in 2012 the extreme rays of C_5 and these DHD span those extreme rays that Laurent and Vargas could not handle ...

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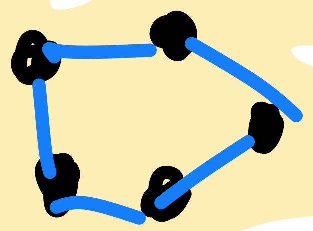
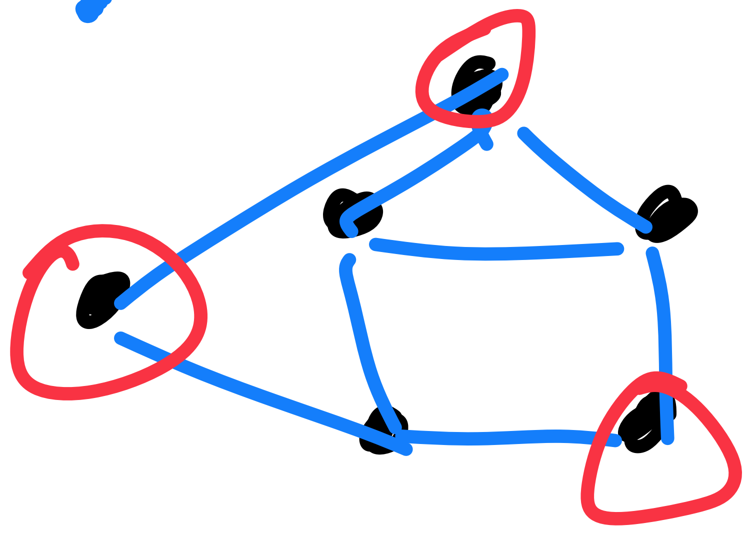
$H = M$ 

Computing the stability number of a graph is NP-hard.



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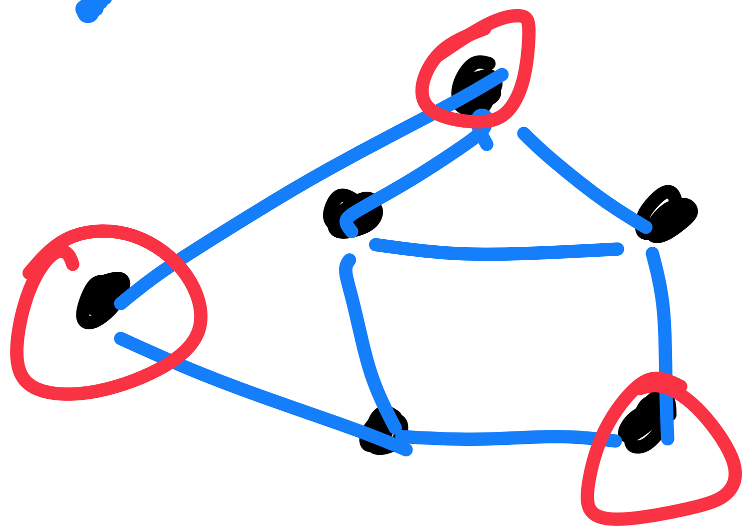

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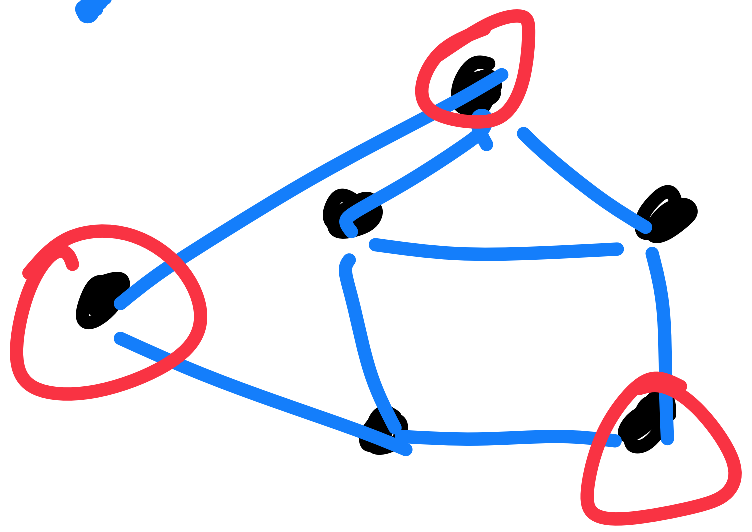

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We show at least finite convergence:

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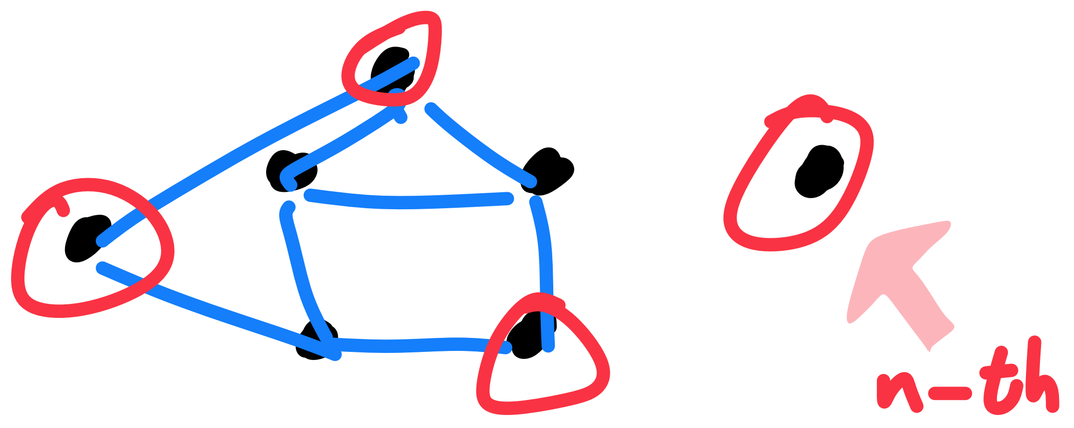

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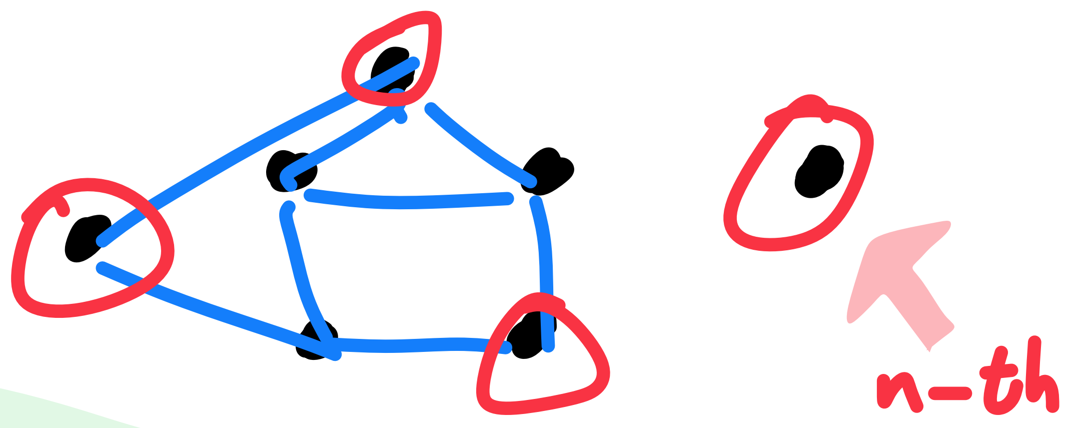

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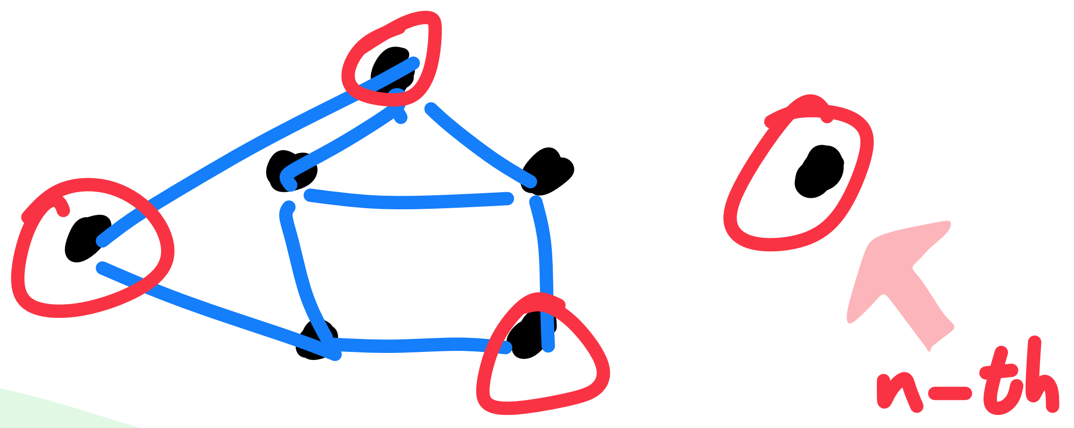

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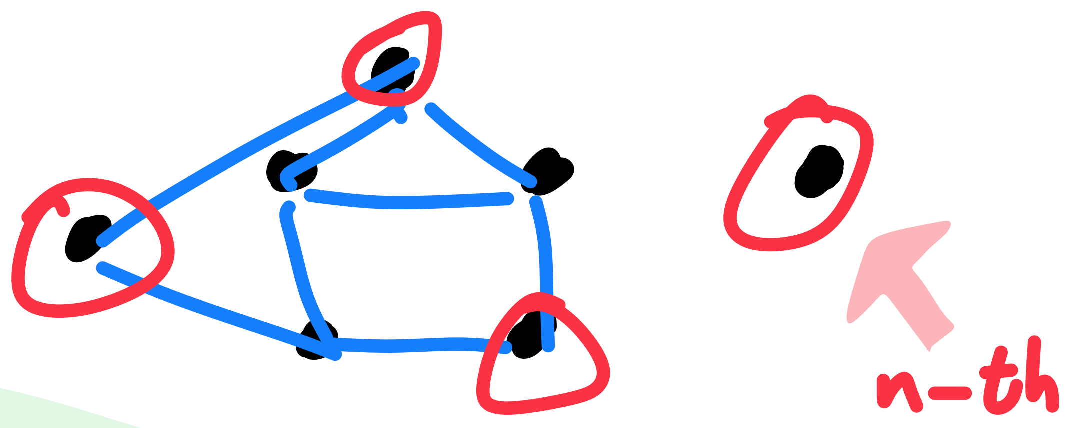

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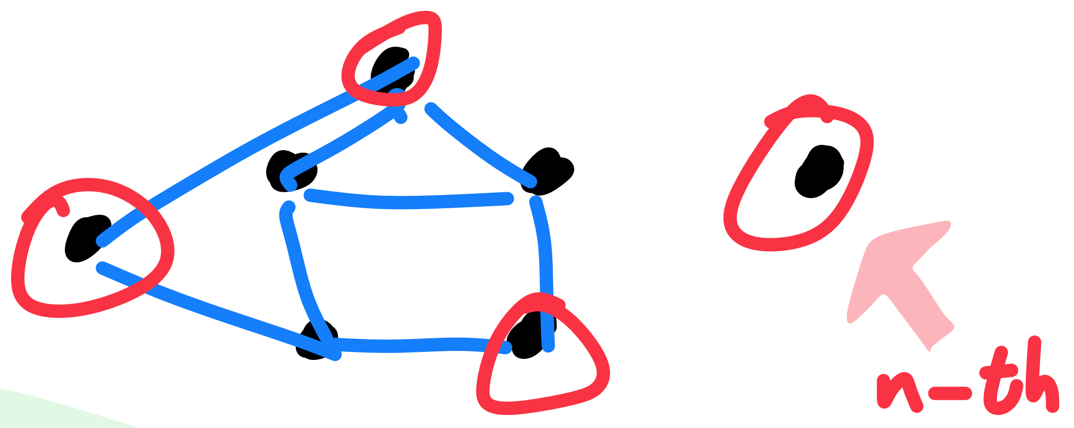

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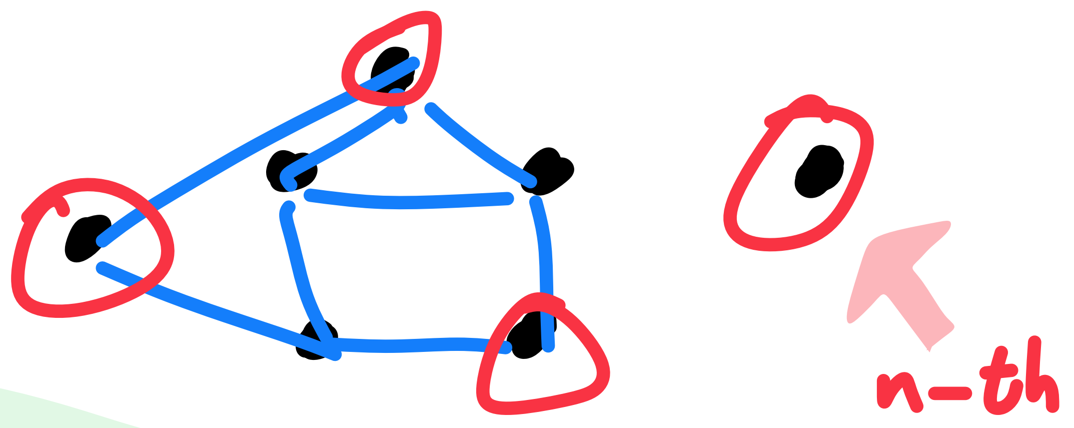

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$$h := (x_1^2 \dots x_n^2) M_H \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}.$$

Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \dots + x_{n-1}^2)^r g \in \sum \mathbb{R}[x]^2$

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It turns out that $h = p^2 + cg$ for some $p \in \mathbb{R}[x]$ and $c > 0$.

Theorem 11 Let $F \subseteq \mathbb{R}[x]$ generate the ideal I. Let M be an Archimedean quadratic module of $\mathbb{R}[x]$ and $f, u \in I$. Suppose that

(a) $f \geq 0$ on $S(M)$

(b) $\forall a \in S(M) : (f(a) = 0 \implies u(a) = 0)$

(c) $uM \subseteq M$

(d) u is F-stably contained in M, i.e., $\forall f \in F : \exists \varepsilon > 0 : u \pm \varepsilon f \in M$

(e) $\varphi(f) > 0$ for all zeros a of f on $S(M)$ and all test states φ on I for M at a wrt. u.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

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$f := h$, $M := M_{S^{n-1}}$

Step 1. $u := p^2 + c(x_1^2 + \ldots + x_{n-1}^2)^{2r} g \in \sum \mathbb{R}[x]^2$

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(e) $\varphi(f) > 0$ for all zeros a of f on $S(M)$ and all test states φ on I for M at a wrt. u.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

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Step 1. $u := p^2 + c(x_1^2 + \ldots + x_{n-1}^2)^{2r} g \in \sum \mathbb{R}[x]^2$

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Step 3. Let φ be a test state on $I := \mathbb{R}[x]p^2 + \mathbb{R}[x]g$ for M at an S^{n-1} wrt. u.

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(e) $\varphi(f) > 0$ for all zeros a of f on $S(M)$ and all test states φ on I for M at a wrt. u.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon u \in M$. In particular, $f \in M$.

$f := h$, $M := M_{S^{n-1}}$

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Step 3. Let φ be a test state on $I := \mathbb{R}[x]p^2 + \mathbb{R}[x]g$ for M at a zero a of f on S^{n-1} wrt. u.

Then $1 = \varphi(u) = \varphi(p^2) + c(a_1^2 + \ldots + a_{n-1}^2)^{2r} \varphi(g)$.

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(c) $vM \subseteq M$

(d) v is F-stably contained in M, i.e., $\forall f \in F : \exists \varepsilon > 0 : v \pm \varepsilon f \in M$

(e) $\varphi(f) > 0$ for all zeros a of f on $S(M)$ and all test states φ on I for M at a wrt. v.

Then, there is $\varepsilon > 0$ such that $f - \varepsilon v \in M$. In particular, $f \in M$.

$f := h$, $M := M_{S^{n-1}}$

Step 1. $v := p^2 + c(x_1^2 + \dots + x_{n-1}^2)^{2r} g \in \sum \mathbb{R}[x]^2$

Step 2. $F := \{p^2, g\}$ (very tricky, two pages)

Step 3. Let φ be a test state on $I := \mathbb{R}[x]p^2 + \mathbb{R}[x]g$ for M at a zero a of f on S^{n-1} wrt. v.

Then $1 = \varphi(v) = \underbrace{\varphi(p^2)} + \underbrace{c(a_1^2 + \dots + a_{n-1}^2)^{2r}}_{\geq 0} \underbrace{\varphi(g)}$.

at least one of these two positive!

$$h := (x_1^2 \dots x_n^2) M_H \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}.$$ Choose $r \in \mathbb{N}_0$ such that $(x_1^2 + \dots + x_{n-1}^2)^r g \in \sum \mathbb{R}[x]^2$

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Moreover $0 \leq \varphi((x_1^2 + \ldots + x_{n-1}^2)^r g) = \underbrace{(a_1^2 + \ldots + a_{n-1}^2)^r}_{> 0} \varphi(g)$.

since $a \neq (0, \ldots, 0, \pm 1)$ because $f(a) = 0$ (isolated node!)

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Moreover $0 \leq \varphi((x_1^2 + \ldots + x_{n-1}^2)^r g) = \underbrace{(a_1^2 + \ldots + a_{n-1}^2)^r}_{>0} \varphi(g)$. Hence $\varphi(p^2) \geq 0$ and $\varphi(g) \geq 0$.

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Finally, $\varphi(f) = \varphi(h) = \varphi(p^2) + c\,\varphi(g) > 0$.

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