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Real Algebraic Geometry II – Exercise Sheet 7

Exercise 1 (4P) Let K be a subfield of $\mathbb{R}$, V a finite-dimensional K-vector space, $A \subseteq V$ convex and F a maximal nontrivial face of A. Show that F is exposed.

Exercise 2 (8P) Let $n \in \mathbb{N}_0$ and $d \in \mathbb{N}$ be even, $V \subseteq \mathbb{R}[X_1, \dots, X_n]$ the $\mathbb{R}$-vector space of all d-forms in n variables. Let $P \subseteq V$ be the cone of all positive semidefinite d-forms of n variables. Show:

(a) P is closed.

(b) P° consists exactly of the positive definite d-forms in n variables.

(c) For every $x \in \mathbb{R}^n \setminus \{0\}$ the set $F_x := \{f \in P \mid f(x) = 0\}$ is a maximal non-trivial face of P.

(d) For every maximal non-trivial face F of V there exists an $x \in \mathbb{R}^n \setminus \{0\}$ such that $F = F_x$.

Exercise 3 (8P) Suppose K is a subfield of $\mathbb{R}$, $n \in \mathbb{N}_0$ and V is an n-dimensional topological K-vector space. Let $A \subseteq V$ be a convex set and $x \in V \setminus A$. Show that there exist K-linear functions $\varphi_1, \dots, \varphi_n \colon V \to \mathbb{R}$ such that for every $y \in A$, there exists $j \in \{1, \dots, n\}$ satisfying

$$\varphi_1(x) = \varphi_1(y), \dots, \varphi_{j-1}(x) = \varphi_{j-1}(y) \text{ and } \varphi_j(x) < \varphi_j(y).$$

Exercise 4 (4P)

(a) Prove or disprove the following: For any $n \in \mathbb{N}_0$ and closed $A \subseteq \mathbb{R}^n$, $\operatorname{conv}(A)$ is also closed.

(b) Find $n \in \mathbb{N}$ and two nonempty disjoint convex sets $A, B \subseteq \mathbb{R}^n$ such that there exists no linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\varphi(a) < \varphi(b)$ for all $a \in A$ and $b \in B$.

Please submit until Tuesday, June 13, 2017, 9:55 in the box named RAG II near to the room F411.