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**Winter Term 2016/2017**

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# Real Algebraic Geometry I – Exercise Sheet 1

On this sheet, we will prove, among other things, the existence and uniqueness of the real numbers. You are therefore not allowed to use what (you think) you already know about the real numbers.

**Exercise 1** (4P). Let $(K, \leq)$ be an ordered field and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences in $K$ with $a := \lim_{n\to\infty} a_n$ and $b := \lim_{n\to\infty} b_n$. Show that

$$\lim_{n\to\infty}(a_n + b_n) = a + b \qquad \text{and} \qquad \lim_{n\to\infty} a_n b_n = ab.$$

**Exercise 2** (4P). Let $(K, \leq)$ be an ordered field. Show that the following statements are equivalent:

(a) $(K, \leq)$ is Archimedean and Cauchy-complete.

(b) $(K, \leq)$ is complete.

**Hint:** (a) $\Longrightarrow$ (b): Suppose $A \subseteq K$ is a nonempty subset bounded from above. Choose for every $n \in \mathbb{N}$ the smallest $k_n \in \mathbb{Z}$ such that $\forall a \in A : a \leq \frac{k_n}{n}$ and set $a_n := \frac{k_n}{n} \in \mathbb{Q}$ (use the Archimedean property!). Show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent. Finally, show that $a := \lim_{n\to\infty} a_n$ is the lowest upper bound of $A$ in $(K, \leq)$.

For (b) $\Longrightarrow$ (a) use contraposition. First suppose that $(K, \leq)$ is not Archimedean, that is to say the set $A := \{a \in K \mid \forall N \in \mathbb{N} : a \leq -N\}$ is not empty. We claim that $A$ does not have a lowest upper bound since if $a \in K$ is an upper bound of $A$, one can show that $a - 1$ is again an upper bound of $A$.

Finally, suppose that $(K, \leq)$ is not Cauchy-complete, that is to say there is a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $K$ which does not converge in $K$. Show that then the set $A := \{a \in K \mid \exists N \in \mathbb{N} : \forall n \geq N : a \leq a_n\}$ is not empty and bounded from above, however does not have a lowest upper bound.

**Exercise 3** (4P). Let $(K, \leq)$ be an Archimedean ordered field and $(R, \leq_R)$ a complete ordered field. Show that there is exactly one embedding $(K, \leq) \hookrightarrow (R, \leq_R)$ and that this is an isomorphism if and only if $(K, \leq)$ is complete.

**Exercise 4** (4P). Show that there exists a complete ordered field $(\mathbb{R}, \leq)$ and that it is essentially unique in the following sense: If $(K, \leq_K)$ is another complete ordered field, then there is a unique isomorphism from $(K, \leq_K)$ onto $(\mathbb{R}, \leq)$.

**Hint:** Existence: Show that the Cauchy sequences in $\mathbb{Q}$ form a subring $C$ of $\mathbb{Q}^{\mathbb{N}}$ and that

$$I := \{(a_n)_{n\in\mathbb{N}} \in C \mid \lim_{n\to\infty} a_n = 0\}$$

is a maximal ideal in $C$. Set $\mathbb{R} := C/I$. Show that

$$a \leq b :\iff \exists (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in C : (a = \overline{(a_n)_{n\in\mathbb{N}}}^I \ \&\ b = \overline{(b_n)_{n\in\mathbb{N}}}^I \ \&\ \forall n \in \mathbb{N} : a_n \leq b_n)$$

$(a, b \in \mathbb{R})$ defines an order $\leq$ on $\mathbb{R}$. Clearly, $(\mathbb{R}, \leq)$ is Archimedean. Due to Exercise 2 it remains to show that $(\mathbb{R}, \leq)$ is Cauchy-complete. So take a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}, \leq)$. Using 1.1.10 we find a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ with $|a_n - q_n| < \frac{1}{n}$ for $n \in \mathbb{N}$. Show that since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, \leq)$, also $(q_n)_{n\in\mathbb{N}}$ is such in $(\mathbb{R}, \leq)$ and hence also in $\mathbb{Q}$. Thus $(q_n)_{n\in\mathbb{N}} \in C$. Set $a := \overline{(q_n)_{n\in\mathbb{N}}}^I$. Show $\lim_{n\to\infty} a_n = a$.

**Please submit until Thursday, November 3, 2016, 11:44 in the box named RAG I near to the room F411.**