LICENCE BIOLOGIE MATHÉMATIQUES 2008-2009

Corrigé du premier contrôle continu.

1. $$\frac{\partial f(x,y,z)}{\partial x} = \frac{1}{2+z^4}$$

$$\frac{\partial f(x,y,z)}{\partial y} = \frac{3y^2}{2+z^4}$$

$$\frac{\partial f(x,y,z)}{\partial z} = (x+y^3)\left(-\frac{1}{(2+z^4)^2}\right)4z^3 = \frac{-4(x+y^3)z^3}{(2+z^4)^2}$$

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} = 0$$

$$\frac{\partial^2 f(x,y,z)}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f(x,y,z)}{\partial x \partial z} = \frac{-4z^3}{(2+z^4)^2}$$

$$\frac{\partial^2 f(x,y,z)}{\partial y^2} = \frac{6y}{2+z^4}$$

$$\frac{\partial^2 f(x,y,z)}{\partial y \partial z} = 3y^2\left(-\frac{1}{(2+z^4)^2}\right)4z^3 = \frac{-12y^2z^3}{(2+z^4)^2}$$

$$\frac{\partial^2 f(x,y,z)}{\partial z^2} = \frac{(2+z^4)^2(-12(x+y^3)z^2) + 4(x+y^3)z^3 2(2+z^4)4z^3}{(2+z^4)^4}$$

$$= \frac{(x+y^3)\left(-12z^2(2+z^4)+32z^6\right)}{(2+z^4)^3}$$

$$= \frac{4(x+y^3)z^2(5z^4-6)}{(2+z^4)^3}.$$

Le théorème de Schwarz assure que

$$\frac{\partial^2 f(x,y,z)}{\partial y \partial x} = \frac{\partial^2 f(x,y,z)}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f(x,y,z)}{\partial z \partial x} = \frac{\partial^2 f(x,y,z)}{\partial x \partial z} = \frac{-4z^3}{(2+z^4)^2}$$

$$\frac{\partial^2 f(x,y,z)}{\partial z \partial y} = \frac{\partial^2 f(x,y,z)}{\partial y \partial z} = \frac{-12y^2z^3}{(2+z^4)^2}.$$

2.(a) L'aire d'un rectangle dont les côtés mesurent $a$ et $b$ est

$$A = a \cdot b.$$

On a $\ln(A) = \ln(a) + \ln(b)$, $\frac{\partial \ln(A)}{\partial a} = \frac{1}{a}$, $\frac{\partial \ln(A)}{\partial b} = \frac{1}{b}$ et donc

$$\frac{\Delta A}{A} \sim \Delta \ln(A) \sim \frac{\partial \ln(A)}{\partial a}\Delta a + \frac{\partial \ln(A)}{\partial b}\Delta b$$

$$= \frac{\Delta a}{a} + \frac{\Delta b}{b}.$$

On a donc une borne supérieure approchée pour l'incertitude relative de l'aire de 2 % car

$$\left|\frac{\Delta A}{A}\right| \lessapprox \left|\frac{\Delta a}{a}\right| + \left|\frac{\Delta b}{b}\right| \leq 0{,}01 + 0{,}01 = 0{,}02.$$

(b) Le volume d'un cuboïd dont les côtés mesurent a, b et c est

$$V = abc.$$

On a $\ln(V) = \ln(a) + \ln(b) + \ln(c)$,

$$\frac{\partial \ln(V)}{\partial a} = \frac{1}{a}, \quad \frac{\partial \ln(V)}{\partial b} = \frac{1}{b} \text{ et } \frac{\partial \ln(V)}{\partial c} = \frac{1}{c}$$

d'où

$$\frac{\Delta V}{V} \sim \Delta \ln(V) \sim \frac{\partial \ln(V)}{\partial a}\Delta a + \frac{\partial \ln(V)}{\partial b}\Delta b + \frac{\partial \ln(V)}{\partial c}\Delta c$$

$$= \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c}$$

On a donc une borne supérieure approchée pour l'incertitude relative du volume de 3% car

$$\left|\frac{\Delta V}{V}\right| \lessapprox \left|\frac{\Delta a}{a}\right| + \left|\frac{\Delta b}{b}\right| + \left|\frac{\Delta c}{c}\right| \leq 0{,}01 + 0{,}01 + 0{,}01 = 0{,}03.$$

3. (a) $\int \frac{1}{x}\,dx = \ln|x|$

(b) $\int x^6 \exp(x^7)\,dx \underset{\substack{\frac{dt}{dx} = 7x^6 \\ x^6 dx = \frac{dt}{7}}}{\overset{t = x^7}{=}} \frac{1}{7}\int \exp(t)\,dt$

$= \frac{1}{7}\exp(t) = \frac{1}{7}\exp(x^7)$

(c) $\int \frac{56x^3+8x+2}{7x^4+2x^2+x+1}\,dx \underset{\frac{dt}{dx} = 28x^3+4x+1}{\overset{t = 7x^4+2x^2+x+1}{=}}$

$\int \frac{2dt}{t} = 2\int \frac{1}{t}\,dt = 2\ln|t|$

$= 2\ln|7x^4+2x^2+x+1|$

$= 2\ln(7x^4+2x^2+x+1)$

$\uparrow$

$7x^4+2x^2+x+1 = 7x^4+\frac{3}{2}x^2+\frac{1}{4}+\frac{1}{2}(x+1)^2 \geq 0$ pour tout $x \in \mathbb{R}$

(d) $\int (\sin x)(\cos x)^2\,dx \underset{\substack{\frac{dt}{dx} = -\sin x \\ (\sin x)\,dx = -dt}}{\overset{t = \cos x}{=}} -\int t^2\,dt = -\frac{t^3}{3}$

$= -\frac{(\cos x)^3}{3}$

(e) $$\int \frac{\ln x}{x^2}\,dx = \int \underbrace{\frac{1}{x^2}}_{u'}\underbrace{(\ln x)}_{v}\,dx$$

$$= \underbrace{\frac{x^{-2+1}}{-2+1}}_{u}\underbrace{\ln x}_{v} - \int \underbrace{\frac{x^{-2+1}}{-2+1}}_{u}\,\underbrace{\frac{1}{x}}_{v'}\,dx$$

$$= -\frac{\ln x}{x} + \int x^{-2}\,dx$$

$$= -\frac{\ln x}{x} + \frac{x^{-2+1}}{-2+1} = -\frac{\ln x}{x} - \frac{1}{x}$$

$$= -\frac{1}{x}(1+\ln x) = -\frac{1+\ln x}{x}$$

(f) $$\int \frac{\sqrt{x-1}}{x}\,dx \qquad \begin{array}{l} t=\sqrt{x-1} \\ x-1=t^2 \\ x=1+t^2 \\ \frac{dx}{dt}=2t \end{array} \qquad 2\int \frac{t^2}{1+t^2}\,dt$$

$$\sqrt{x-1}\,dx = 2\sqrt{x-1}\,t\,dt = 2(x-1)\,dt = 2t^2\,dt$$

$$= 2\left(\int \frac{1+t^2}{1+t^2}\,dt - \int \frac{1}{1+t^2}\,dt\right)$$

$$= 2\left(\int dt - \int \frac{1}{1+t^2}\,dt\right)$$

$$= 2(t - \arctan t)$$

$$= 2(\sqrt{x-1} - \arctan\sqrt{x-1})$$